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Stabilization of systems with changing dynamics

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Abstract. We present a framework for designing stable control schemes for systems whose dynamic equations change as they evolve on the state space. It is usually difficult or even impossible to design a single controller that would stabilize such a system. An appealing alternative are switching control schemes, where a different controller is employed on each of the regions defined by different dynamic characteristics and the stability of the overall system is ensured through appropriate switching scheme. We derive sufficient conditions for the stability of a switching control scheme in a form that can be used for controller design. An important feature of the proposed framework is that although the overall hierarchy can be very complicated, the stability depends only on the immediate relation of each controller to its neighbors. This makes the application of our results particularly straight forward. The methodology is applied to stabilization of a shimmying wheel, where changes in the dynamics are due to switches between sliding and rolling.

1 Introduction

The design of controllers for hybrid systems is a difficult problem that is still not satisfactorily solved. Most existing design methodologies assume that the underlying dynamics are continuous and that the hybrid behavior arises because the system must perform several functions. The control synthesis task is then to design controllers that achieve each of the functions and a coordination scheme that guarantees that properties like safety and liveness are satisfied at all times. This work addresses a different problem. We study dynamical systems that change their dynamic behavior as they evolve in the state space. The hybrid nature is thus inherent in the dynamics of the system and does not come from the controller specification. In this paper we study the problem of stabilization of such systems. The goal is to design a controller that stabilizes an equilibrium set in one of the regions, moving through other regions if necessary. We achieve this by designing a controller on each of the regions and a scheme for switching between these controllers. We show that the stability of the overall system can be guaranteed by imposing conditions on controllers that operate on adjacent regions. This leads to modularity of the design process and considerably simplifies the synthesis problem. The stability analysis is based on Lyapunov functions.

A starting point for controller design is a choice of a formalism for description of a hybrid system. In the literature we can find several alternatives. Alur et al. [1] and Nicollin at al. [2] defined the notion of hybrid automaton, building their work on the automata theory. Brockett [3] devised his model using the theory of dynamical systems. Other works in this category are [4] and [5]. Branicky gives an overview of such models and relates them to his own model [6]. We use models in this second group for our work.

Prior work on hybrid controller design has often been limited to specific applications. Lygeros et al. [7] proposed a game-theoretic framework for design of controllers for intelligent highway systems and air traffic control systems. Puri [8] and Deshpande [9] developed methods for controller design using a simplified version of hybrid automata. Kohn et al. developed a methodology for coordination of multiple agents [10]. Branicky & Mitter [11] and Žefran et al. [12] employed optimal control for synthesis of open-loop trajectories. Goodwine & Burdick [13] developed a controllability test and a planning method for a class of hybrid systems called stratified systems. An important step in controller design is verification. The approaches in [7]–[9] include verification as an integral part of the design process. Some other works that address the verification are [14], [15], and [16].

A number of authors considered stability of hybrid controllers. Branicky [17] devised sufficient conditions for stability of a system that switches between different controllers that stabilize an equilibrium point. Based on this work, Malmborg et al. [18] proposed a strategy for choosing a controller among several available controllers so that the overall system is stable. Both papers allow dynamic equations to change, but they are primarily concerned with the case when the equilibrium point is the same for each controller so there is no need to actively drive the system into some designated region, as we do in the present paper. An earlier work on stability of switching controllers is also [19].

The idea of driving the system through a sequence of equilibrium points until a desired equilibrium point is reached was employed in [20]. In this work, the switch between different controllers always occurs at an equilibrium point. The authors also assume that the region of attraction of each controller is known so there is no need for Lyapunov functions to prove the stability.

The paper is organized as follows. We start with a motivating example and introduce some notions for stability analysis on manifolds. We next formulate three propositions that give sufficient conditions for the stability of a switching controller. The propositions are progressively less abstract and lead to a practical synthesis methodology. We then apply the methodology to solve the problem of stabilization for the classical shimmying wheel. We demonstrate the behavior of the controller with some simulation results and conclude the paper with a brief discussion.

2 Preliminaries

To motivate the theoretical development we start with an example. The system that we study is the classical shimmying wheel [21, 22]. A schematic of the shimmying wheel is shown in Fig. 1. A rigid link with a wheel is attached to a hinge joint, which is in turn connected to a rigid object through a sliding joint between two springs (Fig. 1). The control input is the torque at the hinge joint. The object moves with a constant velocity v in the direction perpendicular to the axis of the sliding joint. The shimmying wheel can be seen as a simplified model of an aircraft nose wheel or a motorcycle front wheel, with the springs modeling the compliance of the wheel and the wheel attachment [22]. It can also serve as a model of a vehicle towing a trailer, with the springs abstracting the compliance in the kingpin.

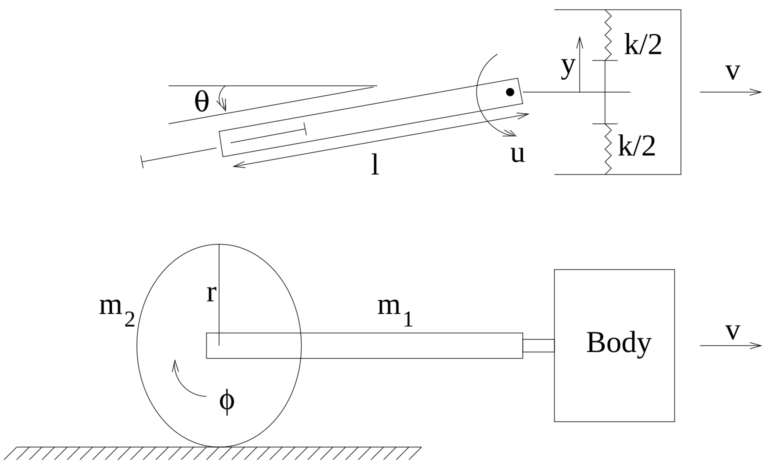

Fig. 1. A top view and a side view of a shimmying wheel.

The goal of control is to stabilize the wheel so that the bar is aligned with the direction of v (perpendicular to the sliding axis) and the slider is in the neutral position between the two springs (the forces of the springs are equal in magnitude and of the opposite sign). This task is complicated by the fact that the system can operate in two regimes: the wheel can either roll without sliding or it can slip. The slipping regime is undesirable, but often unavoidable. The system will switch between rolling and sliding depending on the magnitude of the contact force between the wheel and the ground: the wheel will slip if the force in rolling would be greater than the friction force. If we assume a feedback control law for the torque about the hinge joint, the contact force is completely determined by the state of the system and the state space gets divided into two regions separated by a switching surface on which the contact force equals the friction force. In each of the regions the equations of motion are different. It is therefore unlikely that a single controller could stabilize the system and even if one exists it is not clear how to design it.

A controller that is designed without taking the hybrid nature of the dynamics into account can produce undesired results. It is for example possible to design a stable controller that linearizes the shimmying wheel dynamics if the wheel is rolling. Figure 2(a) shows that this controller efficiently stabilizes the system. However, if the same controller is used while the wheel is sliding, *it can destabilize the system*, as can be seen in Fig. 2(b). This example shows that a more comprehensive approach to design of controllers for systems with hybrid dynamics is needed.

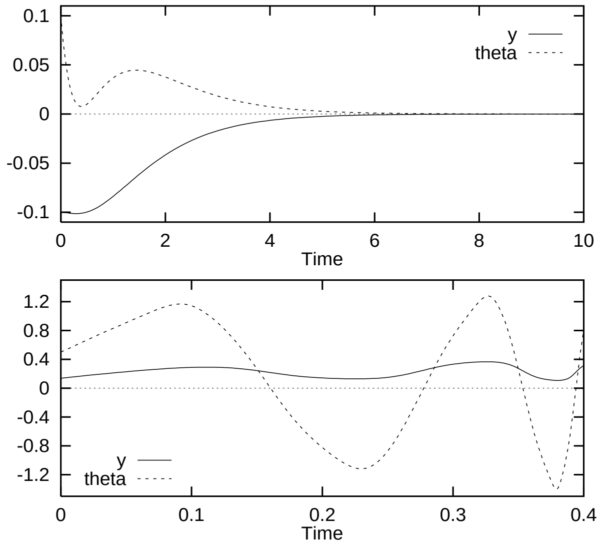

Fig. 2. A linearizing controller applied in rolling (a) and sliding (b).

2.1 Stability theory on manifolds

We are interested in stabilizing submanifolds (possibly unbounded). Conventional Lyapunov theory can not be directly applied to this setting, so we need to introduce some additional concepts (see [23]).

Definition 1. A *distance* between a point x and a set $E \subseteq \mathbb{R}^n$ is defined by:

$$\rho(x, E) = \inf_{y \in E} d(x, y) \tag{1}$$

A *ball* with radius R around E is the set $B(E, R) = \{x \mid \rho(x, E) < R\}$.

Definition 2. A smooth manifold $E \subset M$ is *locally stable* if for any $R > 0$ there exist $r > 0$ such that if $\rho(x(t_0), E) < r$ then $\rho(x(t), E) < R$ for every $t > t_0$. If, in addition, $\lim_{t \to \infty} \rho(x(t), E) = 0$, then we say that E is *locally asymptotically stable.*

Definition 3. A submanifold $E \subset M$ is *locally attractive* if there exists $R > 0$ such that if $\rho(x(t_0), E) < R$ then $\lim_{t\to\infty} \rho(x(t), E) = 0$. We also say that trajectories starting inside $B(E, R)$ *converge* to E.

Theorem 4 [24, 25]. *If for a control system* Σ *there exists a* C^1 *function* $V : M \to \mathbb{R}$*, such that:*

(1) $V(x) \geq 0$ *and* $V(x) = 0 \Leftrightarrow x \in E$*;*
(2) there exists a monotonically increasing function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$, $\alpha(0) = 0$*, such that* $\alpha(\rho(x, E)) < V(x)$*;*
(3) there exists a monotonically increasing function $\beta : \mathbb{R}^+ \to \mathbb{R}^+$, $\beta(0) = 0$*, such that* $V(x) < \beta(\rho(x, E))$*;*
(4) $\dot{V}(x) \leq 0$*, where* $\dot{V}$ *is the derivative of* V *along the trajectories of* Σ*;*

then the manifold E *is locally stable. If in addition:*

(5) there exists a monotonically increasing function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$, $\gamma(0) = 0$*, such that* $\dot{V}(x) \leq -\gamma(\rho(x, E)) < 0$,

then E *is locally asymptotically stable.*

2.2 Modeling

In this section we describe the setting which will be used to formally describe systems whose dynamics change. Suppose we have a dynamical system Σ and a collection of (differentiable, connected) manifolds $\mathcal{M} = \{M_1, M_2, \ldots, M_n\}$. The manifolds need not be disjoint, they can be a subset of each other and in some cases it it will be even convenient to take some of them to be equal. This collection of manifolds must reflect the changing dynamics, but additional manifolds can be defined for the purposes of a particular application. An example of a collection of manifolds is shown in Fig. 3(a). On each manifold, the system is described with a set of equations:

$$\dot{x}_i = f_i(x_i, u_i, t), \tag{2}$$

where x_i is the state of the system and u_i is the vector of inputs for the system evolving on the submanifold M_i. In general, f_i's can be different to reflect changes in the dynamics of the system. Also the dimensions of the manifolds might be different. For example, in the case of the shimmying wheel, the manifolds M_1 and M_2 would correspond to sliding and rolling, respectively, where M_1 is the whole space and M_2 is the subspace on which the rolling constraint is satisfied.

We will assume that on each manifold M_i we design a controller g_i:

$$u_i = g_i(x_i, t), \tag{3}$$

The reason of allowing some manifolds in the collection $\mathcal{M}$ to be the same is that we may wish to define different controllers on the same physical space. Let $E_n \subseteq M_n$ be a manifold to which we wish to steer the system Σ. The problem that we address in this paper is how to design the controllers g_i and a rule for

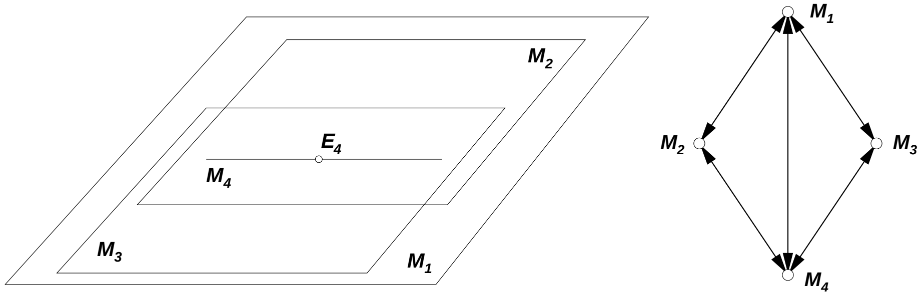

Fig. 3. (a) A sequence of embedded manifolds; (b) the corresponding graph.

switching among them (a switching scheme) that stabilizes the system to E_n (if possible globally). This task is complicated by the fact that, in general, we do not know the sequence of the manifolds that the dynamical system will traverse. Take for example the shimmying wheel. If the system is rolling, the controller action might cause the wheel to slip, but it is conceivable that within a certain region such switching does not happen. And it is of course always possible that a disturbance (for example a slippery patch) causes the rolling wheel to slip.

The topology of a system evolving on a collection of manifolds can be described with a graph. The vertices of the graph correspond to different manifolds. There will be an edge from a manifold M_i to a manifold M_j if it is possible to switch from M_i to M_j (there exists a trajectory that passes from M_i to M_j). For example, if we assume that a nonempty intersection of two manifolds implies that it is possible to pass between the manifolds, the graph for the system in Fig. 3(a) would be Fig. 3(b).

3 Sufficient conditions for stability

Take a control system Σ evolving on the collection of manifolds $\mathcal{M}$. Assume that on each manifold M_i, we have a controller g_i (i.e., $u_i = g_i(x,t)$). Let the controller g_n stabilize the manifold E_n (i.e., the target manifold). Assume we can construct a Lyapunov function V_n which satisfies the conditions (1)-(5) of Theorem 4. Let

$$\begin{aligned} \mathcal{S}: \quad \mathrm{I\!R}^n \times \{1,\ldots,n\} &\to \{1,\ldots,n\} \\ (x,\eta) &\mapsto \mathcal{S}(x,\eta) \end{aligned} \tag{4}$$

denote the switching scheme. In other words, the function $\mathcal{S}$ selects the controller to be used, depending on the state x, and the controller that is currently used, η. Clearly, $\mathcal{S}(x,\eta) = i$ implies $x \in M_i$, since g_i is only defined on M_i. The following proposition gives sufficient conditions for E_n to be globally attractive:

Proposition 5. *Let the switching scheme $\mathcal{S}$ satisfy the following conditions:*

1. There exists $L > 0$ such that $\mathcal{S}(x, n) = n$ for every $x \in B(E_n, L) \cap M_n$.
2. For any trajectory $x(t)$ there exists a $\Delta > 0$ and an infinite sequence $\{t_i\}$ whose elements satisfy:
   (a) for every $t \in [t_i, t_i + \Delta]$, $\mathcal{S}(x(t), \eta(t)) = n$;
   (b) $V(t_i + \Delta) \geq V(t_{i+1})$.

Then the submanifold E_n is globally attractive.

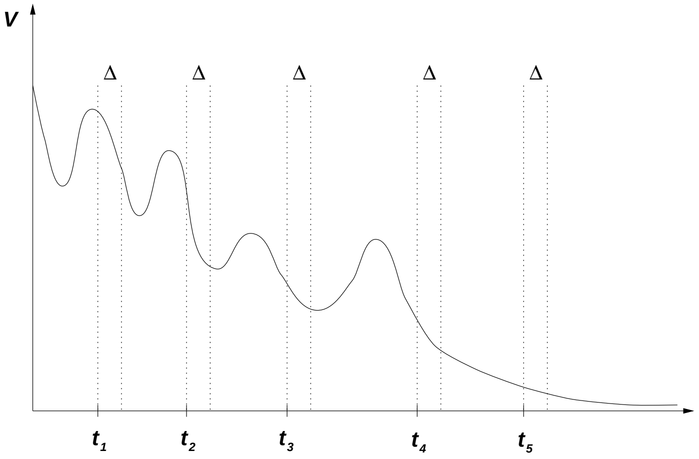

Fig. 4. Values of the Lyapunov function and a sequence satisfying condition 2(b) of Proposition 5.

Remark: Condition (1) guarantees that there is a region around E_n in which it is not possible to switch from g_n to some other controller g_i. That is, we assume that the controller g_n can capture and stabilize Σ in some region around E_n. Condition (2) states that regardless of the current state, the system will eventually come under the control of g_n and stay under the control of g_n for at least time Δ. Furthermore, we can find a sequence of time subintervals of length at least Δ so that the Lyapunov function restricted to the union of these intervals is monotonically decreasing.

Proof. Let $\{t_k\}$ be a sequence given by condition (2). Since the Lyapunov function V is monotonically decreasing when the system evolves on M_n, condition 2(b) implies that $t_{i+1} - t_i \geq \Delta$. Now take $I_n = \cup_{k \in \mathbb{N}} [t_k, t_k + \Delta]$ and consider the system evolving on I_n. By assumption, V satisfies the conditions of Theorem 4, so we can find monotonically increasing functions α, β and γ such that $\alpha(\rho(x, E_n)) < V(x) < \beta(\rho(x, E_n))$ and $\dot{V}(x) \leq -\gamma(\rho(x, E_n)) < 0$. Let $r = \rho(x(t_0), E_n)$ and let ϵ be an arbitrary number such that $0 < \epsilon < r$. Then we can find $\delta > 0$ such that $\beta(\delta) < \alpha(\epsilon)$. Let K be an integer such that $K > \frac{\beta(r)}{\Delta\gamma(\delta)}$

and take $\tau = t_K + \Delta$. Suppose that $\rho(x(t), E_n) > \epsilon$ for every $t \in I_n \cap [t_0, \tau]$. Then we have:

$$
\begin{aligned}
0 < \alpha(\epsilon) &\leq V(x(\tau)) = V(x(t_K)) + \int_{t_K}^{t_K+\Delta} \dot{V}(x(t))\, dt \\
&\leq V(x(t_K)) - \int_{t_K}^{t_K+\Delta} \gamma(\rho(x(t), E_n))\, dt \leq V(x(t_K)) - \int_{t_K}^{t_K+\Delta} \gamma(\delta)\, dt \\
&= V(x(t_K)) - \Delta\gamma(\delta) \leq V(x(t_{K-1})) - \Delta\gamma(\delta) \leq \dots \\
&\leq V(x(t_0)) - K\Delta\gamma(\delta) \leq \beta(r) - K\Delta\gamma(\delta) < 0 \qquad (5)
\end{aligned}
$$

This is a contradiction, implying that there exists $\tau' \in I_n \cap [t_0, \tau]$ such that $\rho(x(\tau'), E_n) < \delta$. But then for every $t \in I_n$ such that $t > \tau'$:

$$\alpha(\rho(x(t), E_n)) \leq V(x(t)) \leq V(x(\tau')) \leq \beta(\delta) < \alpha(\epsilon)$$

which implies:

$$\rho(x(t), E_n) < \epsilon \;\; \forall t > \tau', t \in I_n$$

This shows that $\rho(x(t), E_n)$ converges to 0 on I_n.

Since $\rho(x(t), E_n)$ converges to 0 on I_n, there exists $T > 0$ such that for all $t > T, t \in I_n$, $\rho(x(t), E_n) < L$. But by assumption, for $x \in B(E_n, L) \cap M_n$ the system can not switch from M_n to some M_j, $j \neq i$, which means that the system will stay under the control of g_n for all $t > T$ and therefore converge to E_n.

While the lemma provides sufficient conditions for convergence of the system trajectories to E_n, these conditions are difficult to check and therefore not suitable for controller design. It is particularly difficult to check condition (2). We therefore provide two additional tests that are less general, but are easier to apply.

Take $M_1, M_2, \dots, M_n$, the collection of manifolds on which a dynamical system evolves, and let $A = \{1, 2, \dots, n\}$ be the index set. The switching scheme $\mathcal{S}$ defines a relation Switch(A), if we put Switch(i, j) when it is possible to switch from the manifold M_i (controller g_i) to the manifold M_j (controller g_j). More formally:

$$\text{Switch}(A) = \{(i, j) \mid \exists x \in M_i \;\text{ s.t. }\; \mathcal{S}(x, i) = j\} \qquad (6)$$

Note that the graph representing this relation is precisely the graph described in Section 2.2. We can then show:

Proposition 6. *Let $\preceq$ be a partial order within the transitive closure of the relation* Switch(A) *which has the smallest element, and let this smallest element be n. Assume that the switching scheme $\mathcal{S}$ has the following properties:*

1. *There exists $L > 0$ such that $\mathcal{S}(x, n) = n$ for every $x \in B(E_n, L) \cap M_n$.*
2. *If $x(t)$ is a trajectory of Σ and M_i, $i \neq n$ is a manifold on which $x(t)$ evolves for an infinite amount of time, then there exists $\Delta > 0$ such that for every T we can find $\tau > T$ such that $\mathcal{S}(x(t), \eta(t)) \prec i$ for every $t \in [\tau, \tau + \Delta]$.*
3. *If a system switched from g_n to some other controller at time t_{off} and if t_{on} is the time when the system next switches again to g_n, then $V(t_{\text{off}}) \geq V(t_{\text{on}})$.*

Then the submanifold E_n is globally attractive.

Remark: The first condition is the same as in Proposition 5, while conditions (2) and (3) together replace condition (2) there. Condition (2) says that for any manifold M_i on which a trajectory stays for an infinite amount of time, we can find a switch at an arbitrary large time to a manifold that lies lower in the hierarchy implied by $\preceq$ and that after such switch the system evolves on the manifolds that are below M_i for at least Δ.

Proof. We will show that conditions (2) and (3) imply condition (2) of Proposition 5. Let $x(t)$ be a trajectory of Σ and let M_i be a manifold on which $x(t)$ evolves for an infinite amount of time. Since we have a finite number of manifolds, there will be at least one such i. The condition (2) guarantees that there will be an infinite number of instances when the system evolves for at least Δ on manifolds that are below M_i in the hierarchy defined by $\preceq$. But this implies that $x(t)$ will evolve on these manifolds for an infinite amount of time and since there are only finitely many manifolds below M_i, there must exist a manifold M_j with $j \prec i$ on which $x(t)$ evolves for an infinite amount of time. By proceeding recursively and because n is the smallest element for $\preceq$, we conclude that the system must evolve on M_n for an infinite amount of time and in instances that last for at least Δ. Condition (3) guarantees that each time the system switches to g_n, the value of the Lyapunov function is smaller than when the system last switched off M_n. The existence of the sequence $\{t_i\}$ in condition (2) of Proposition 5 is therefore guaranteed.

Using Proposition 6 we can design a stable switching scheme by choosing a partial order, developing controllers on each M_i that guarantee a switch to a lower level with respect to this partial order, and enforcing decreasing of V at switches to M_n. However, developing controllers that guarantee a switch to a lower level is still not an easy task. One possible strategy is to make each controller stabilize a certain manifold within a region from which the system switches to manifolds lower in the hierarchy. This special case is important enough that we state a separate proposition.

Proposition 7. *Assume a partial order $\preceq$ on A that has the smallest element which is equal to n. Let each controller g_i asymptotically stabilize a manifold E_i and assume we can find a Lyapunov function V_i for g_i. Let the switching scheme $\mathcal{S}$ satisfy the following conditions:*

1. *For each i, there exists $L_i > 0$ such that $\mathcal{S}(x,i) \prec i$ for every $x \in B(E_i, L_i) \cap M_i$ (for $i = n$ we require $\mathcal{S}(x,n) = n$).*
2. *There exists $\Delta > 0$, such that if a system switches from g_i to some g_j, $j \prec i$ at time T, then $\mathcal{S}(x(t), \eta(t)) \prec i$ for each $t \in [T, T+\Delta]$.*
3. *If the system switches from g_i to some g_j, $i \prec j$, at time t_{off} and after that switches again to g_i at time t_{on} and if $\mathcal{S}(x(t), \eta(t)) \not\prec i$ for all $t \in [t_{\text{off}}, t_{\text{on}}]$, then $V_i(t_{\text{off}}) \geq V_i(t_{\text{on}})$.*

Then the submanifold E_n is globally attractive.

Remark: For $i = n$ conditions (1) and (3) above clearly become the same as conditions (1) and (3) in Proposition 6. Note that the Proposition suggests that we can examine the stability of the system by simply examining relations between neighbors defined by the switching scheme. This has important implications for the synthesis problem and can be explored to obtain modularity of the design process.

Proof. We will show that the above conditions imply conditions of the Proposition 6. Assume that a trajectory $x(t)$ evolves on a manifold M_i for an infinite amount of time, but after some time T it never switches to any manifold M_j such that $j \prec i$. Let $I_i = \{t > T \mid \mathcal{S}(x(t), \eta(t)) = i\}$, the union of the intervals beyond T during which the system evolves on M_i. By condition (3), V_i will be monotonically decreasing on I_i and by condition (2), we can find an infinite sequence of (disjoint) intervals of length Δ that lie in I_i. By the same reasoning that we used in the proof of Proposition 5 to show convergence to E_n we can show that $x(t)$ converges to E_i. By condition (1) this implies that the system will switch to some M_j, $j \prec i$, which is a contradiction. This and condition (2) above therefore imply condition (2) of Proposition 6.

The last proposition is a convenient tool for designing stable switching control schemes. The algorithm for controller design can be roughly described as:

- Choose a partial order on A (decide on the hierarchy among M_i's).
- Design a controller on each M_i that stabilizes a manifold E_i.
- Choose a neighborhood U_i of E_i and define a switching scheme so that for $x \in U_i$, $\mathcal{S}(x, i) \prec i$.

Clearly, this basic algorithm has to be refined to guarantee that the conditions (2) and (3) above are satisfied.

There is an important case in which condition (2) can be satisfied fairly easily. Suppose we want to switch from M_i to M_j, $j \prec i$. If $f_j(x, g_j(x))$ in Eq. (2) is bounded for all $x \in U \subset M_j$, where U is a neighborhood that contains the region to which the system switches, then all we need to do is make the system switch in such a way that after the switch to M_j we are some (fixed) finite distance away from any point x in U for which $j \prec \mathcal{S}(x, j)$. Because of the bounded rate of change of the state, this implies that the switch will occur after some finite time interval.

It is difficult to directly design controllers that would satisfy condition (3). An alternative is to combine several controllers, each of which partly satisfies the condition, into a single controller. Suppose we would like to allow switches from M_i to M_j, $j \prec i$. To satisfy condition (3), we need to have a controller g_i that is able to decrease the Lyapunov function V_j. Controller g_i stabilizes E_i, and we also know that the controller g_j decreases the Lyapunov function V_j. If E_i is the equilibrium manifold for the system controlled by g_i, we can construct a new controller, $\hat{g}_i$ that behaves as g_i away from E_i and as g_j close to E_i. A possible expression for $\hat{g}_i$ would be:

$$\hat{g}_i(x) = (1 - c_1 e^{-c_2\, d(x, E_1)}) g_i(x) + c_1 e^{-c_2\, d(x, E_1)} g_j(x) \tag{7}$$

where c_1 and c_2 are appropriate constants.

Propositions 5-7 provide sufficient conditions for E_n to be attractive, not to be stable. To prove the stability we have to show that trajectories starting outside M_n “nicely” converge to M_n. One possible way of stating this is:

Corollary 8. *The manifold E_n will be stable if in addition to the conditions of Proposition 5:*

(3) For any $R > 0$ and every i, there exists $r > 0$ such that if $x(t_0) \in (M_i \setminus M_n) \cap B(E_n, r)$ then under the control of g_i, $x(t) \in B(E_n, R)$ for every $t > t_0$.

Proof. The Lyapunov function V guarantees that for any $R_2 > 0$, there exists $r_2 > 0$ such that $x(t_0) \in M_n \cap B(E_2, r_2)$ implies $x(t) \in B(E_n, R_2)$ as long as $x(t)$ stays in M_n. Take $R_2 = \min\{R, L\}$ and find the corresponding r_2. Take $R_1 = \min\{R, r_2\}$. By assumption, there exists r_1 such that $x(t)$ stays in $B(E_n, R_1)$ for any trajectory starting in $B(E_n, r_1) \setminus M_n$ and evolving in M_i. By condition (1) of Proposition 5 and by the choice of R_1, $x(t)$ will intersect M_n inside $B(E_n, r_2) \cap M_n$. But a trajectory on M_n that comes inside $B(E_n, r_2) \cap M_n$ will stay inside $B(E_n, L) \cap M_n$ and thus remain under the control of g_n (and stay inside $B(E_n, R)$) for all later times.

We note that this proof is similar to the proof of Theorem 4 in [17].

Remark If we assume the scenario of Proposition 7 and for every i, $E_i \subseteq E_n$, the condition of the Corollary will be trivially true.

4 Example

The above results provide a framework for designing hybrid control schemes. In this section we apply the methodology to stabilization of the shimmying wheel (Fig. 1). Dynamic equations of the system are of the form:

$$H \begin{bmatrix} \ddot{y} \\ \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} + \begin{bmatrix} ky + \frac{l}{2}(m_1 + 2m_2)\dot{\theta}^2 \sin\theta \\ 0 \\ 0 \end{bmatrix} = A^T F + \begin{bmatrix} 0 \\ u \\ 0 \end{bmatrix} \tag{8}$$

where H is the inertia matrix, $F = \{F_x, F_y\}^T$ is the reaction force of the ground on the wheel, and A is the matrix that relates the relative velocity v_r between the wheel and the ground at the contact point to the rate of change of the generalized coordinates. The system has 6 states: 3 generalized coordinates and 3 generalized velocities.

When the wheel is sliding, we have the following expression for the reaction force $F = F_s$:

$$F_s = -\mu_d \frac{v_r}{\|v_r\|}(m_1 + \frac{m_2}{2})g \tag{9}$$

where μ_d is the coefficient of (dynamic) friction and g is the gravity constant. When the wheel is rolling, we have an additional constraint:

$$v_r = 0 \tag{10}$$

In this case, the force $F = F_c$ is the constraint force that prevents slippage of the wheel and it can be eliminated from Eq. (8) using Eq. (10) [21, 22]. Equation (10) represents two constraint equations, so the dimension of the system in pure rolling drops to 4. The analysis of the system can be simplified by observing that ϕ does not occur in the dynamic equations. It is therefore a cyclic variable and we can limit our study to the dynamics of y and θ. In the formalism of Section 2, the reduced system thus evolves on manifolds M_1 and M_2 of dimension 4 and 3, respectively, where $M_1 = \mathbb{R}^4$ and M_2 is defined by Eq. (10) [21, 22].

The goal of the control is to stabilize the wheel to the state $y = 0$, $\theta = 0$. To this end, we introduce an additional region, M_3, but we put $M_3 = M_2$. In other words, we use two different controllers in the rolling regime. Note that nothing in the developed theory prohibits the submanifolds to be equal. Stabilization is therefore achieved with three controllers: a controller g_1 for the system in sliding regime (defined on M_1) and controllers g_2 and g_3 for the system in the rolling mode (defined on M_2). The idea is to steer the system with the controllers g_1 and g_2 to a state $\theta = 0$, $y \neq 0$, from which we can stabilize the system to a desired point with the controller g_3. Note that the wheel might start sliding again once under the control of g_3.

To design a controller for the system evolving on M_1, we linearize the dynamic response for θ. It can be shown that with this controller the dynamics for y and ϕ are also (asymptotically) stable. The controller stabilizes the line segment:

$$E_1 = (y, 0, 0, 0) \qquad |y| \leq \frac{\mu_d(m_1 + 2m_2)g}{2k}$$

The controller g_2 (only defined on M_2, when the wheel is rolling) can be designed similarly to g_1 after the constraint force is eliminated from dynamic equations using Eq. (10). The attractive manifold for this controller is a line:

$$E_2 = (y, \theta, \dot{y}, \dot{\theta}) = (y, 0, 0, 0)$$

The controller g_3 can be derived by observing that instead of the dynamics for θ, we can linearize the dynamics for y. Further analysis shows that with this controller, the dynamics for θ and ϕ are stable, so the system converges to the desired point, $E_3 = (0, 0, 0, 0)$. It is also not difficult to construct the Lyapunov functions V_2 and V_3 for the controllers g_2 and g_3.

Next, we have to define a partial order and design the switching schemes. We first observe that there is a natural partial order already defined on $\mathcal{M} = \{M_1, M_2, M_3\}$ and it is given by inclusion: $M_1 \supset M_2 \supseteq M_3$. The partial order in this case thus becomes a total order and the application of Proposition 7 is therefore particularly straight forward.

The switching scheme $\mathcal{S}_1$ is quite simple:

$$\mathcal{S}_1(x,\eta) = \begin{cases} 2 & x \in M_2 \ \wedge \ \|F_c\| \le \frac{\mu_d}{2}(m_1 + 2m_2)g \\ 1 & \text{otherwise} \end{cases}$$

The controller g_2 has a singularity at $\theta = \pm\frac{\pi}{2}$, but on these two hyperplanes the constraint force is unbounded and they do not intersect (the closure of) M_2. The switching scheme $\mathcal{S}_2$ is defined in the following way:

$$\mathcal{S}_2(x,\eta) = \begin{cases} 3 & \eta = 2 \ \wedge \ x \in B(E_3, R_{\text{in}}) \ \wedge \ V_3(x) \le V_3^{3\to 2} \\ & \quad \wedge \ \|F_c\| \le \frac{\mu_s}{4}(m_1 + 2m_2)g \\ 3 & \eta = 3 \ \wedge \ x \in B(E_3, R_{\text{out}}) \\ 2 & \text{otherwise} \end{cases}$$

where $R_{\text{in}} < R_{\text{out}} < \frac{\pi}{2}$ (this guarantees that $B(E_3, R_{\text{out}})$ does not intersect the hyperplanes $\theta = \pm\frac{\pi}{2}$), and $V_3^{3\to 2}$ is the value of V_3 when the system last switched from the controller g_3 to the controller g_2. Again, we avoid the hyperplanes $\theta = \pm\frac{\pi}{2}$ because g_3 becomes singular there. Observe that the switching scheme explicitly encodes condition (3) of Proposition 7.

The next step would be to check that the conditions of the Proposition 7 are satisfied. Since we have a total order on $\mathcal{M}$, it suffices to show that g_1 and g_2 stabilize E_2, and that g_2 and g_3 stabilize E_3. In the interest of keeping the presentation short the proofs will be omitted, but we refer the interested reader to [26] for details. We only mention that in order to guarantee that the controller g_2 can arbitrarily decrease the Lyapunov function V_3 so that the system can switch to g_3, we use the technique described in Eq. (7).

4.1 Simulation results

A typical simulation run of the system controlled with the derived controllers is shown in Fig. 5. The system starts in the sliding regime with the controller g_1 active. At 0.9s the wheel stops sliding and the controller g_2 takes over. At 1.14s the system switches again, this time to the controller g_3 that stabilizes the system to the desired state. The switches between different controllers cause discontinuities of the input, as Fig. 5(b). shows. It can be seen in Fig. 5(a) that while the controllers g_1 and g_2 are active, θ is the controlled variable and it decreases to 0. When the controller g_3 becomes active, the controlled variable becomes y (so it decreases to 0) and $|\theta|$ initially increases. After y becomes small, $|\theta|$ also decreases to 0.

The next figure illustrates that the modified controller $\hat{g}_2$ decreases the Lyapunov function V_3. Variables y and θ are shown in Fig. 6(a), while the Lyapunov functions V_2 and V_3 are shown in Fig. 6(b). The system starts in the rolling regime with the controller g_3 active, however during the first 0.1s it switches first to the controller g_2 and then to the sliding regime and the controller g_1 (these switches are not shown). At the switch from g_3 to g_2 the value of the Lyapunov function V_3 is 263.4. To show that the controller can arbitrary decrease

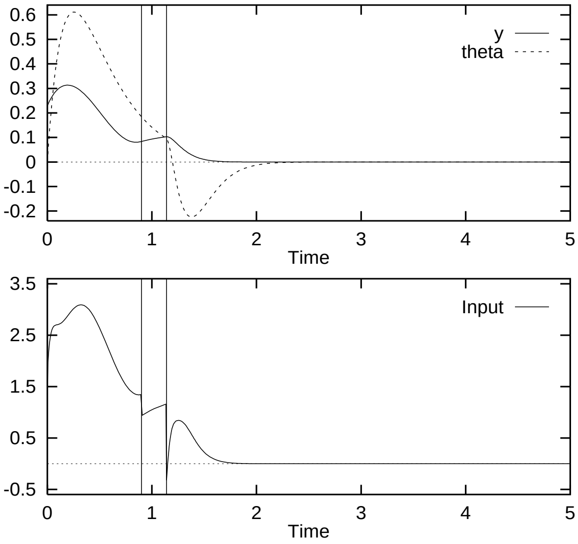

Fig. 5. A typical simulation run.

V_3, we modified the switching scheme $\mathcal{S}_2$ so that the value of the Lyapunov function V_3 at the switch from g_2 to g_3 has to be half the value of the function at the switch from g_3 to g_2. In our case, the function V_3 therefore has to decrease to 131.7 in order to switch to the controller g_3. At the time 0.38s, the system switches from sliding to rolling and to the controller g_2. The controller decreases the Lyapunov function until it reaches the desired value at the time 1.30s when the system switches to the controller g_3 and the system is stabilized. Figure 6(a) also shows that the controller g_2 does not drive θ to 0 but to some offset value that guarantees the decreasing of V_3.

5 Conclusion

We investigated the problem of stabilizing a system with changing dynamics with a sequence of controllers. We studied the case when the system evolves on a sequence of embedded manifolds and derived sufficient conditions under which the switching scheme employing different controllers can be guaranteed to stabilize the system to the desired manifold. These sufficient conditions give direct guidance for the design of appropriate controllers. The results were applied to the stabilization of the shimmying wheel. We were able to design a switching scheme that provably stabilizes this system.

The described work can be extended in several directions. We plan to consider more general stabilization problems such as control of a walking robot. In this case, the system has to be stabilized to a periodic orbit that traverses different

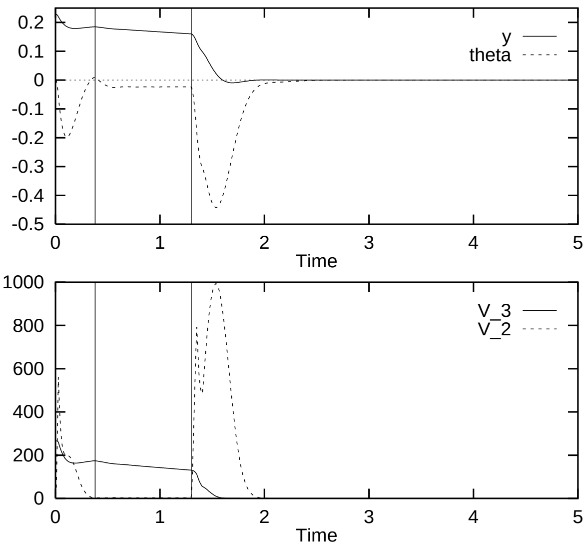

Fig. 6. A modified controller guarantees decreasing of V_3.

regions rather than an equilibrium manifold within a single region. An important question is also how to design the individual controllers. For mechanical systems, the energy-momentum method offers some interesting possibilities.

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