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Chapter 1
Lagrangian Dynamics

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The motion of a mechanical system is related via a set of *dynamic equations* to the forces and torques it is subject to. In this work we will be primarily interested in robots consisting of a collection of rigid links connected through joints that constrain the relative motion between the links. There are two main formalisms for deriving the dynamic equations for such mechanical systems: Newton-Euler equations that are directly based on Newton's laws; and Euler-Lagrange equations that have their root in the classical work of d'Alembert and Lagrange on analytical mechanics and the work of Euler and Hamilton on variational calculus. The main difference between the two approaches is in dealing with constraints. While Newton's equations treat each rigid body separately and explicitly model the constraints through the forces required to enforce them, Lagrange and d'Alembert provided systematic procedures for eliminating the constraints from the dynamic equations, typically yielding a simpler system of equations. Constraints imposed by joints and by other mechanical components are one of the defining features of robots so it is not surprising the Lagrange's formalism is often the method of choice in robotics literature.

1.1 Preliminaries

The approach and the notation in this section are inspired by [21] and we refer the reader to that text for a additional details. A starting point in describing a physical system is the formalism for describing its motion. Since we will be concerned with robots consisting of rigid links, we start by describing rigid body motion. Formally, a rigid body $\mathcal{O}$ is a subset of $\mathbb{R}^3$ where each element in $\mathcal{O}$ corresponds to a point on the rigid body. The defining property of a rigid body is that the distance between arbitrary two points on the rigid body remains unchanged as the rigid body moves. If a *body-fixed* coordinate frame B is attached to $\mathcal{O}$, an arbitrary point $p \in \mathcal{O}$ can be described by a fixed vector p_{B}. As a result, the position of any point on $\mathcal{O}$ is uniquely determined by the location of the frame B. To describe the location of B in space we choose a global coordinate frame S. The position and orientation of the frame B in the frame S is called the *configuration* of $\mathcal{O}$ and can be described by a

4×4 *homogeneous matrix* g_{SB}:

$$g_{\mathrm{SB}} = \begin{bmatrix} R_{\mathrm{SB}} & d_{\mathrm{SB}} \\ 0 & 1 \end{bmatrix}, R_{\mathrm{SB}} \in \mathbb{R}^{3\times 3}, d_{\mathrm{SB}} \in \mathbb{R}^3, R_{\mathrm{SB}}^T R_{\mathrm{SB}} = I_3, \det(R_{\mathrm{SB}}) = 1. \tag{1.1}$$

Here I_n denotes the identity matrix in $\mathbb{R}^{n\times n}$. The set of all possible configurations of $\mathcal{O}$ is known as $SE(3)$, the special Euclidean group of rigid body transformations in three dimensions. By the above argument, $SE(3)$ is equivalent to the set of homogeneous matrices. It can be shown that it is a matrix group as well as a smooth manifold and, therefore, a *Lie group*; for more details we refer to [11, 21]. It is convenient to denote matrices $g \in SE(3)$ by the pair (R, d) for $R \in SO(3)$ and $d \in \mathbb{R}^3$.

Given a point $p \in \mathcal{O}$ described by a vector p_{B} in the frame B, it is natural to ask what is the corresponding vector in the frame S. From the definition of g_{SB} we have

$$\overline{p}_{\mathrm{S}} = g_{\mathrm{SB}} \overline{p}_{\mathrm{B}}$$

where for a vector $p \in \mathbb{R}^3$, the corresponding *homogeneous vector* $\overline{p}$ is defined as:

$$\overline{p} = \begin{bmatrix} p \\ 1 \end{bmatrix}.$$

The tangent space of $SE(3)$ at $g_0 \in SE(3)$ is the vector space of matrices of the form $\dot{g}(0)$, where $g(t)$ is a curve in $SE(3)$ such that $g(0) = g_0$. The tangent space of a Lie group at the group identity is called the *Lie algebra* of the Lie group. The Lie algebra of $SE(3)$, denoted by $se(3)$, is

$$se(3) = \left\{ \begin{bmatrix} \Omega & v \\ 0 & 0 \end{bmatrix} \mid \Omega \in \mathbb{R}^{3\times 3}, v \in \mathbb{R}^3, \Omega^T = -\Omega \right\}. \tag{1.2}$$

Elements of $se(3)$ are called *twists*. Recall that a 3×3 skew-symmetric matrix Ω can be uniquely identified with a vector $\omega \in \mathbb{R}^3$ so that for an arbitrary vector $x \in \mathbb{R}^3$, $\Omega x = \omega \times x$, where $\times$ is the vector cross product operation in $\mathbb{R}^3$. Each element $T \in se(3)$ can be thus identified with a 6×1 vector

$$\xi = \begin{bmatrix} v \\ \omega \end{bmatrix},$$

called the *twist coordinates* of T. We also write $\Omega = \widehat{\omega}$ and $T = \widehat{\xi}$ to denote these transitions between vectors and matrices.

An important relation between the Lie group and its Lie algebra is provided by the exponential map. It can be shown that for $\widehat{\xi} \in se(3)$, $\exp(\widehat{\xi}) \in SE(3)$, where $\exp : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ is the usual matrix exponential. Using the properties of $SE(3)$ it can be shown that, for $\xi^T = \begin{bmatrix} v^T & \omega^T \end{bmatrix}$,

$$\exp(\widehat{\xi}) = \begin{cases} \begin{bmatrix} I_3 & v \\ 0 & 1 \end{bmatrix}, & \omega = 0, \\ \begin{bmatrix} \exp(\widehat{\omega}) & \frac{1}{\|\omega\|^2}\left[(I_3 - \exp(\widehat{\omega}))(\omega \times v) + \omega\omega^T v\right] \\ 0 & 1 \end{bmatrix}, & \omega \neq 0, \end{cases} \tag{1.3}$$

where the relation

$$\exp(\widehat{\omega}) = I_3 + \frac{\sin\|\omega\|}{\|\omega\|}\,\widehat{\omega} + \frac{1-\cos\|\omega\|}{\|\omega\|^2}\,\widehat{\omega}^2,$$

is known as *Rodrigues' formula*. From the last formula it is easy to see that the exponential map $\exp : se(3) \to SE(3)$ is many to one. It can be shown that the map is in fact **onto**. In other words, every matrix $g \in SE(3)$ can be written as $g = \exp(\widehat{\xi})$ for some $\widehat{\xi} \in se(3)$. The components of ξ are also called *exponential coordinates* of g.

To every twist $\xi^T = \left[\begin{array}{cc} v^T & \omega^T \end{array}\right]$, $\omega \neq 0$, we can associate a triple (l, h, M) called the *screw associated with* $\widehat{\xi}$, where we define $l = \{p + \lambda\omega \in \mathbb{R}^3 |\ p \in \mathbb{R}^3, \lambda \in \mathbb{R}\}$, $h \in \mathbb{R}$, and $M \in \mathbb{R}$ so that the following relations hold:

$$M = \|\omega\|, \qquad v = -\omega \times p + h\omega.$$

Note that l is the line in $\mathbb{R}^3$ in the direction of ω passing through the point $p \in \mathbb{R}^3$. If $\omega = 0$, the corresponding screw is $(l, \infty, \|v\|)$, where $l = \{\lambda v \in \mathbb{R}^3 |\ \lambda \in \mathbb{R}\}$. In this way, the exponential map can be given an interesting geometric interpretation: if $g = \exp(\widehat{\xi})$ with $\omega \neq 0$, then the rigid body transformation represented by g can be realized as a rotation around the line l by the angle M followed by a translation in the direction of this line for the distance hM. If $\omega = 0$ and $v \neq 0$, the rigid body transformation is simply a translation in the direction of $\frac{v}{\|v\|}$ for a distance M. The line l is the *axis*, h is the *pitch* and M is the *magnitude* of the screw. This geometric interpretation and the fact that every element $g \in SE(3)$ can be written as the exponential of a twist is the essence of *Chasles Theorem.*

1.1.1 Velocities and forces

We have seen that $SE(3)$ is the configuration space of a rigid body $\mathcal{O}$. By considering a rigid body moving in space, a more intuitive interpretation can be also given to $se(3)$. At every instant t, the configuration of $\mathcal{O}$ is given by $g_{\mathrm{SB}}(t) \in SE(3)$. The map $t \mapsto g_{\mathrm{SB}}(t)$ is therefore a curve representing the motion of the rigid body. The time derivative $\dot{g}_{\mathrm{SB}} = \frac{d}{dt} g_{\mathrm{SB}}$ corresponds to the velocity of the rigid body motion. However, the matrix curve $t \mapsto \dot{g}_{\mathrm{SB}}$ does not allow an easy geometric interpretation. Instead, it is not difficult to show that the matrices $\widehat{V}^b_{\mathrm{SB}} = g^{-1}_{\mathrm{SB}}\dot{g}_{\mathrm{SB}}$ and $\widehat{V}^s_{\mathrm{SB}} = \dot{g}_{\mathrm{SB}} g^{-1}_{\mathrm{SB}}$ take values in $se(3)$. The matrices $\widehat{V}^b_{\mathrm{SB}}$ and $\widehat{V}^s_{\mathrm{SB}}$ are called the *body* and *spatial velocity*, respectively, of the rigid body motion. Their twist coordinates will be denoted by V^b_{SB} and V^s_{SB}. Assume a point $p \in \mathcal{O}$ has coordinates p_{S} and p_{B} in the spatial frame and in the body frame, respectively. As the rigid body moves along the trajectory $t \mapsto g_{\mathrm{SB}}(t)$, a direct computation shows that the velocity of the point can be computed by $\dot{\overline{p}}_{\mathrm{S}} = \widehat{V}^s_{\mathrm{SB}}\overline{p}_{\mathrm{S}}$ or, alternatively, $\dot{\overline{p}}_{\mathrm{B}} = \widehat{V}^b_{\mathrm{SB}}\overline{p}_{\mathrm{B}}$.

The body and spatial velocities are related by

$$V^s_{\mathrm{SB}} = \mathrm{Ad}_{g_{\mathrm{SB}}} V^b_{\mathrm{SB}}$$

where for $g = (R, d) \in SE(3)$, the *adjoint transformation* $\mathrm{Ad}_g : \mathbb{R}^6 \to \mathbb{R}^6$ is

$$\mathrm{Ad}_g = \left[\begin{array}{cc} R & \widehat{d}R \\ 0 & R \end{array}\right].$$

In general, if $\widehat{\xi} \in se(3)$ is a twist with twist coordinates $\xi \in \mathbb{R}^6$, then for any $g \in SE(3)$ the twist $g\widehat{\xi}g^{-1}$ has twist coordinates $\mathrm{Ad}_g\, \xi \in \mathbb{R}^6$.

Quite often it is necessary to relate the velocity computed in one set of frames to that computed with respect to a different set of frames. Let X, Y, and Z be three coordinate frames. Let g_{XY} and g_{YZ} be the homogeneous matrices relating the Y to the X frame and the Z to the Y frame, respectively. Let $V^s_{\mathrm{XY}} = \dot{g}_{\mathrm{XY}} g^{-1}_{\mathrm{XY}}$ and $V^b_{\mathrm{XY}} = g^{-1}_{\mathrm{XY}} \dot{g}_{\mathrm{XY}}$ be the spatial velocity and the body velocity of the frame Y with respect to the frame X, respectively; define V^s_{XZ}, V^s_{YZ}, V^b_{XZ}, and V^b_{XZ} analogously. The following relations can be verified by a direct calculation:

$$
\begin{aligned}
V^s_{\mathrm{XZ}} &= V^s_{\mathrm{XY}} + \mathrm{Ad}_{g_{\mathrm{XY}}} V^s_{\mathrm{YZ}}, \\
V^b_{\mathrm{XZ}} &= \mathrm{Ad}^{-1}_{g_{\mathrm{YZ}}} V^b_{\mathrm{XY}} + V^b_{\mathrm{YZ}}.
\end{aligned} \tag{1.4}
$$

In the last equation, Ad^{-1} is the inverse of the adjoint transformation and can be computed using the formula $\mathrm{Ad}^{-1}_g = \mathrm{Ad}_{g^{-1}}$, for $g \in SE(3)$.

The body and spatial velocities are in general time dependent. Take the spatial velocity $V^s_{\mathrm{SB}}(t)$ at time t and consider the screw (l, h, M) associated with it. We can show that at this time instant t, the rigid body is moving with the same velocity as if it were rotating around the screw axis l with a constant rotational velocity of magnitude M, and translating along this axis with a constant translational velocity of magnitude hM. If $h = \infty$, then the motion is a pure translation in the direction of l with velocity M. A similar interpretation can be given to $V^s_{\mathrm{SB}}(t)$.

The above arguments show that elements of the Lie algebra $se(3)$ can be interpreted as generalized velocities. It tuns out that the elements of its dual $se^*(3)$, known as *wrenches*, can be interpreted as generalized forces. The fact that the names twist and wrench which originally derived from the screw calculus [5] are used for elements of the Lie algebra and its dual is not just a coincidence, the present treatment can be viewed as an alternative interpretation of the screw calculus.

Using coordinates dual to the twist coordinates, a wrench can be written as a pair $F = \begin{bmatrix} f & \tau \end{bmatrix}$, where f is the force component and τ the torque component. Given a twist $V^T = \begin{bmatrix} v^T & \omega^T \end{bmatrix}$, for $v, \omega \in \mathbb{R}^{3\times 1}$, and a wrench $F = \begin{bmatrix} f & \tau \end{bmatrix}$, for $f, \tau \in \mathbb{R}^{1\times 3}$, the *natural pairing* $\langle W; F\rangle = fv + \tau\omega$ is a scalar representing the *instantaneous work* performed by the generalized force F on a rigid body that is instantaneously moving with the generalized velocity V. In computing the instantaneous work, it is important to consider generalized force and velocity with respect to the same coordinate frame.

As was the case with the twists, we can also associate a screw to a wrench. Given a wrench $F = \begin{bmatrix} f & \tau \end{bmatrix}$, $f \neq 0$, the associated screw (l, h, M) is given by $l = \{p + \lambda f \in \mathbb{R}^3 |\ p \in \mathbb{R}^3, \lambda \in \mathbb{R}\}$ and by:

$$
M = \|f\|, \qquad \tau = -f \times p + hf.
$$

For $f = 0$, the screw is $(l, \infty, \|\tau\|)$, where $l = \{\lambda\tau \in \mathbb{R}^3 |\ \lambda \in \mathbb{R}\}$. Geometrically, a wrench with the associated screw (l, h, M) corresponds to a force with magnitude M applied in the direction of the screw axis l and a torque with magnitude hM applied about this axis. If $f = 0$ and $\tau \neq 0$, then the wrench is a pure torque of magnitude M applied around l. The fact that every wrench can be interpreted in this way is known as *Poinsot Theorem*. Note the similarity with Chasles Theorem.

1.1.2 Kinematics of serial linkages

A serial linkage consists of a sequence of rigid bodies connected through one degree-of-freedom (DOF) revolute or prismatic joints; a multi-DOF joint can be modeled as a sequence of individual 1-DOF joints. To uniquely describe the position of such a serial linkage it is only necessary to know the *joint variables*, i.e., the angle of rotation for a revolute joint or the linear displacement for a prismatic joint. Typically it is then of interest to know the configuration of the links as a function of these joint variables. One of the links is typically designated as the *end-effector*, the link that carries the tool involved in the robot task. For serial manipulators, the end-effector is the last link of the mechanism. The map from the joint space to the end-effector configuration space is known as the *forward kinematics* map. We will use the term *extended forward kinematics map* to denote the map from the joint space to the Cartesian product of the configuration spaces of each of the links.

The configuration spaces of a revolute and of a prismatic joint are connected subsets of the unit circle S^1 and of the real line $\mathbb{R}$, respectively. For simplicity, we shall assume that these connected subsets are S^1 and $\mathbb{R}$, respectively. If a mechanism with n joints has r revolute and $p = n - r$ prismatic joints, its configuration space is $Q = S^r \times \mathbb{R}^p$, where $S^r = \underbrace{S^1 \times \cdots \times S^1}_{r}$. Assume the coordinate frames attached to the individual links are $\mathrm{B}_1, \ldots, \mathrm{B}_n$. The configuration space of each of the links is $SE(3)$, so the extended forward kinematics map is:

$$\begin{array}{rcccl} \kappa & : & Q & \to & SE^n(3) \\ & & q & \mapsto & (g_{\mathrm{SB}_1}(q), \ldots, g_{\mathrm{SB}_n}(q)) \,. \end{array} \tag{1.5}$$

We shall call $SE^n(3)$ the Cartesian space.

For serial linkages the forward kinematics map has a particularly revealing form. Choose a *reference configuration* of the manipulator, i.e. the configuration where all the joint variables are 0, and let $\xi_1, \ldots, \xi_n$, be the *joint twists* in this configuration expressed in the global coordinate frame S. If the ith joint is revolute and $l = \{p + \lambda\omega \in \mathbb{R}^3 | \ \lambda \in \mathbb{R}, \|\omega\| = 1\}$ is the axis of rotation, then the joint twist corresponds to the screw $(l, 0, 1)$ and is:

$$\xi_i = \begin{bmatrix} -\omega \times p \\ \omega \end{bmatrix}. \tag{1.6}$$

If the ith joint is prismatic and $v \in \mathbb{R}^3$, $\|v\| = 1$, is the direction in which it moves, then the joint twist corresponds to the screw $(\{\lambda v \in \mathbb{R}^3 | \ \lambda \in \mathbb{R}\}, \infty, 1)$ and is:

$$\xi_i = \begin{bmatrix} v \\ 0 \end{bmatrix}.$$

One can show that

$$g_{\mathrm{SB}_i}(q) = \exp(\xi_1 q^1) \cdots \exp(\xi_i q^i) g_{\mathrm{SB}_i,0} \tag{1.7}$$

where $g_{\mathrm{SB}_i,0} \in SE(3)$ is the reference configuration of the ith link. This is known as the *Product of Exponentials Formula* and was introduced by Brockett [7].

Another important relation is that between the joint rates and the link body and spatial velocities. A direct computation shows that along a curve $t \mapsto q(t)$,

$$V^s_{\mathrm{SB}_i}(t) = J^s_{\mathrm{SB}_i}(q(t))\dot{q}(t)$$

where $J^s_{\mathrm{SB}_i}$ is a configuration-dependent $6 \times n$ matrix, called the *spatial manipulator Jacobian*, that for serial linkages equals

$$J^s_{\mathrm{SB}_i}(q) = \left[\ \xi_{\mathrm{S},1}(q) \quad \ldots \quad \xi_{\mathrm{S},i}(q) \quad 0 \quad \ldots 0\ \right].$$

Here the jth column $\xi_{\mathrm{S},j}(q)$ of the spatial manipulator Jacobian is the jth joint twist expressed in the global reference frame S after the manipulator has moved to the configuration described by q. It can be computed from the forward kinematics map using the formula:

$$\xi_{\mathrm{S},j}(q) = \mathrm{Ad}_{(\exp(\xi_1 q^1) \cdots \exp(\xi_{j-1} q^{j-1}))}\, \xi_j.$$

Similar expressions can be obtained for the body velocity

$$V^b_{\mathrm{SB}_i}(t) = J^b_{\mathrm{SB}_i}(q(t))\dot{q}(t), \tag{1.8}$$

where $J^b_{\mathrm{SB}_i}$ is a configuration-dependent $6 \times n$ *body manipulator Jacobian* matrix. For serial linkages,

$$J^b_{\mathrm{SB}_i}(q) = \left[\ \xi_{\mathrm{B}_i,1}(q) \quad \ldots \quad \xi_{\mathrm{B}_i,i}(q) \quad 0 \quad \ldots \quad 0\ \right], \tag{1.9}$$

where the jth column $\xi_{\mathrm{B}_i,j}(q), j \leq i$ is the jth joint twist expressed in the link frame B_i after the manipulator has moved to the configuration described by q and is given by:

$$\xi_{\mathrm{B}_i,j}(q) = \mathrm{Ad}^{-1}_{(\exp(\xi_j q^j) \cdots \exp(\xi_i q^i) g_{\mathrm{SB}_i,0})}\, \xi_j.$$

1.2 Dynamic equations

In this section we continue our study of mechanisms composed of rigid links connected through prismatic or revolute joints. One way to describe a system of interconnected rigid bodies is to describe each of the bodies independently and then explicitly model the joints between them through the constraint forces. Since the configuration space of a rigid body is $SE(3)$, a robot manipulator with n links would be described with $6n$ parameters. Newton's equations can be then directly used to describe robot dynamics. An alternative is to use *generalized coordinates* and describe just the degrees of freedom of the mechanism. As discussed earlier, for a robot manipulator composed of n links connected through 1-DOF joints the generalized coordinates can be chosen to be the joint variables. In this case n parameters are therefore sufficient. The Lagrange-d'Alembert Principle can then be used to derive the Euler-Lagrange equations describing the dynamics of the mechanism in generalized coordinates. Because of the dramatic reduction in the number of parameters describing the system this approach is often preferable. We shall describe this approach in some detail.

The Euler-Lagrange equations are derived directly from the energy expressed in the generalized coordinates. Let q be the vector of generalized coordinates. We first form the system *Lagrangian* as the difference between the kinetic and the potential energies of the system. For typical manipulators the Lagrangian function is

$$L(q, \dot{q}) = T(q, \dot{q}) - V(q),$$

where $T(q, \dot{q})$ is the kinetic energy and $V(q)$ the potential energy of the system. The *Euler-Lagrange* equations describing the dynamics for each of the generalized coordinates are then

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = Y_i, \tag{1.10}$$

where Y_i is the *generalized force* corresponding to the generalized coordinate q^i. The subject of the following discussion is study these equations in the context of a single rigid body and of a rigid linkage.

1.2.1 Inertial properties of a rigid body

In order to apply the Lagrange's formalism to a rigid body $\mathcal{O}$ we need to compute its Lagrangian function. Assume that the body-fixed frame B is attached to the center of mass of the rigid body. The kinetic energy of the rigid body is the sum of the kinetic energy of all its particles. Therefore, it can be computed as

$$T = \int_{\mathcal{O}} \frac{1}{2}\|\dot{p}\|^2 dm = \int_{\mathcal{O}} \frac{1}{2}\|\dot{p_{\mathrm{S}}}\|^2 \rho(p_{\mathrm{S}}) dV$$

where ρ is the body density and dV is the volume element in $\mathbb{R}^3$. If $(R_{\mathrm{SB}}, d_{\mathrm{SB}}) \in SE(3)$ is the rigid body configuration, the equality $p_{\mathrm{S}} = R_{\mathrm{SB}} p_{\mathrm{B}} + d_{\mathrm{SB}}$ and some manipulations lead to

$$T = \frac{1}{2} m \|\dot{d}_{\mathrm{SB}}\|^2 + \frac{1}{2}(\omega_{\mathrm{SB}}^b)^T \left(- \int_{\mathcal{O}} \widehat{p}_{\mathrm{B}}^2 \rho(p_{\mathrm{B}}) dV \right) \omega_{\mathrm{SB}}^b \tag{1.11}$$

where m is the total mass of the rigid body and $\widehat{p}_{\mathrm{B}}$ is the skew-symmetric matrix corresponding to the vector p_{B}. The formula shows that the kinetic energy is the sum of two terms referred to as the translational and the rotational components. The quantity $\mathcal{I} = - \int_{\mathcal{O}} \widehat{p_{\mathrm{B}}}^2 \rho(p_{\mathrm{B}}) dV$ is the *inertia tensor* of the rigid body; one can show that $\mathcal{I}$ is a symmetric positive-definite 3×3 matrix. By defining the *generalized inertia matrix*

$$\mathcal{M} = \begin{bmatrix} mI_3 & 0 \\ 0 & \mathcal{I} \end{bmatrix},$$

the kinetic energy can be written as

$$T = \frac{1}{2} \left(V_{\mathrm{SB}}^b\right)^T \mathcal{M} V_{\mathrm{SB}}^b, \tag{1.12}$$

where V_{SB}^b is the body velocity of the rigid body.

Equation (1.12) can be used to obtain the expression for the generalized inertia matrix when the body-fixed frame is not at the center of mass. Assume that $g_{\mathrm{AB}} = (R_{\mathrm{A}}, d_{\mathrm{A}}) \in SE(3)$ is the transformation between the frame A attached to the center of mass of the rigid body and the body-fixed frame B. According to equation (1.4), $V_{\mathrm{SA}}^b = \mathrm{Ad}_{g_{\mathrm{AB}}} V_{\mathrm{SB}}^b$. We thus have

$$T = \frac{1}{2}(V_{\mathrm{SA}}^b)^T \mathcal{M}_{\mathrm{A}} V_{\mathrm{SA}}^b = \frac{1}{2} \left(\mathrm{Ad}_{g_{\mathrm{AB}}} V_{\mathrm{SB}}^b\right)^T \mathcal{M} \left(\mathrm{Ad}_{g_{\mathrm{AB}}} V_{\mathrm{SB}}^b\right) = \left(V_{\mathrm{SB}}^b\right)^T \mathcal{M}_{\mathrm{B}} V_{\mathrm{SB}}^b,$$

where $\mathcal{M}_\mathrm{B}$ is the generalized inertia matrix with respect to the body-fixed frame B:

$$\mathcal{M}_\mathrm{B} = \left(\mathrm{Ad}_{g_\mathrm{AB}}\right)^T \mathcal{M}_\mathrm{A} \, \mathrm{Ad}_{g_\mathrm{AB}} = \begin{bmatrix} mI_3 & mR_\mathrm{A}^T \widehat{d}_\mathrm{A} R_\mathrm{A} \\ -mR_\mathrm{A}^T \widehat{d}_\mathrm{A} R_\mathrm{A} & R_\mathrm{A}^T \left(\mathcal{I} - m\widehat{d}_\mathrm{A}^2\right) R_\mathrm{A} \end{bmatrix}.$$

By observing that $g_\mathrm{BA} = g_\mathrm{AB}^{-1} = (R_\mathrm{B}, d_\mathrm{B})$, where $R_\mathrm{B} = R_\mathrm{A}^T$ and $d_\mathrm{B} = -R_\mathrm{A}^T d_\mathrm{A}$, $\mathcal{M}_\mathrm{B}$ can be written as:

$$\mathcal{M}_\mathrm{B} = \begin{bmatrix} mI_3 & -m\widehat{d}_\mathrm{B} \\ m\widehat{d}_\mathrm{B} & R_\mathrm{B}\mathcal{I}R_\mathrm{B}^T - \widehat{d}_\mathrm{B}^2 \end{bmatrix}.$$

1.2.2 Euler-Lagrange equations for rigid linkages

To obtain the expression for the kinetic energy of a linkage composed of n rigid bodies, we need to add the kinetic energy of each of the links:

$$T = \sum_{i=1}^n T_i = \frac{1}{2}\sum_{i=1}^n (V_{\mathrm{SB}_i}^b)^T \mathcal{M}_{\mathrm{B}_i} V_{\mathrm{SB}_i}^b.$$

Using the relation (1.8), this becomes

$$T = \sum_{i=1}^n \frac{1}{2}(J_{\mathrm{SB}_i}^b \dot{q})^T \mathcal{M}_{\mathrm{B}_i} J_{\mathrm{SB}_i}^b \dot{q} = \frac{1}{2}\dot{q}^T M(q)\dot{q},$$

where

$$M(q) = \sum_{i=1}^n (J_{\mathrm{SB}_i}^b)^T \mathcal{M}_{\mathrm{B}_i} J_{\mathrm{SB}_i}^b$$

is the *manipulator inertia matrix*. For serial manipulators, the body manipulator Jacobian $J_{\mathrm{SB}_i}^b$ is given by equation (1.9).

The potential energy of the linkage typically consists of the sum of the gravitational potential energies of each of the links. Let $h_i(q)$ denote the height of the center of mass of the ith link. The potential energy of the link is then $V_i(q) = m_i g h_i(q)$, and the potential energy of the linkage is:

$$V(q) = \sum_{i=1}^n m_i g h_i(q).$$

If other conservative forces act on the manipulator, the corresponding potential energy can be simply added to V.

The Lagrangian for the manipulator is the difference between the kinetic and potential energies, that is,

$$L(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} - V(q) = \frac{1}{2}\sum_{i,j=1}^n M_{ij}(q)\dot{q}^i\dot{q}^j - V(q)$$

where $M_{ij}(q)$ is the component (i,j) of the manipulator inertia matrix at q. Substituting these expressions into the Euler-Lagrange equations (1.10) we obtain:

$$\sum_{j=1}^{n} M_{ij}(q)\ddot{q}^j + \sum_{j,k=1}^{n} \Gamma_{ijk}(q)\dot{q}^j\dot{q}^k + \frac{\partial V}{\partial q^i}(q) = Y_i, \qquad i \in \{1,\ldots,n\},$$

where the functions Γ_{ijk} are the *Christoffel symbols (of the first kind)* of the inertia matrix M and are defined by

$$\Gamma_{ijk}(q) = \frac{1}{2}\left(\frac{\partial M_{ij}(q)}{\partial q^k} + \frac{\partial M_{ik}(q)}{\partial q^j} - \frac{\partial M_{jk}(q)}{\partial q^i}\right). \tag{1.13}$$

Collecting the equations in a vector format, we obtain

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = Y, \tag{1.14}$$

where $C(q,\dot{q})$ is the *Coriolis matrix* for the manipulator with components

$$C_{ij}(q,\dot{q}) = \sum_{k=1}^{n} \Gamma_{ijk}(q)\dot{q}^k,$$

and where $G_i(q) = \frac{\partial V}{\partial q^i}(q)$. Equation (1.14) suggests that the robot dynamics consists of four components: the inertial forces $M(q)\ddot{q}$, the *Coriolis and centrifugal forces* $C(q,\dot{q})\dot{q}$, the conservative forces $G(q)$ and the generalized force Y composed of all the non-conservative external forces acting on the manipulator. The Coriolis and centrifugal forces depend quadratically on the generalized velocities $\dot{q}^i$ and reflect the inter-link dynamic interactions. Traditionally, terms involving products $\dot{q}^i\dot{q}^j, \;\; i \neq j$ are called Coriolis forces, while centrifugal forces have terms of the form $(\dot{q}^i)^2$.

A direct calculation can be used to show that the robot dynamics has the following two properties [29]. For all $q \in Q$

(i) the manipulator inertia matrix $M(q)$ is symmetric and positive-definite, and

(ii) the matrix $\left(\dot{M}(q) - 2C(q,\dot{q})\right) \in \mathbb{R}^{n\times n}$ is skew-symmetric.

The first property is a mathematical statement of the following fact: the kinetic energy of a system is a quadratic form which is positive unless the system is at rest. The second property is referred to as the *passivity property* of rigid linkages; this property implies that the total energy of the system is conserved in the absence of friction. The property plays an important role in stability analysis of many robot control schemes.

1.2.3 Generalized force computation

Given a configuration manifold Q, the tangent bundle is the set $TQ = \{(q,\dot{q}) |\; q \in Q\}$. The Lagrangian is formally a real-valued map on the tangent bundle, $L : TQ \to \mathbb{R}$. Recall that, given a scalar function on a vector space, its partial derivative is a map from the dual space to the reals. Similarly, it is possible to interpret the partial derivatives $\frac{\partial L}{\partial q}$ and $\frac{\partial L}{\partial \dot{q}}$ as functions taking values in the dual of the tangent bundle, T^*Q. Accordingly, the Euler-Lagrange equations in vector form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = Y$$

can be interpreted as an equality on T^*Q. The external force Y is thus formally a one-form, i.e., a map $Y : Q \to T^*Q$. We let Y_i denote the ith component of Y. Roughly speaking, Y_i is the component of generalized force Y that directly affects the coordinate q^i. In what follows we derive an expression for the generalized force Y_i as a function of the wrenches acting on the individual links.

Let us start by introducing two useful concepts. Recall that given two manifolds Q_1 and Q_2 and a smooth map $\phi : Q_1 \to Q_2$, the *tangent map* $T\phi : TQ_1 \to TQ_2$ is a linear function that maps tangent vectors from T_qQ_1 to $T_{\phi(q)}Q_2$. If $X \in T_qQ_1$ is a tangent vector and $\gamma : \mathbb{R} \to Q_1$ is a smooth curve tangent to X at q, then $T\phi(X)$ is the vector tangent to the curve $\phi \circ \gamma$ at $\phi(q)$. In coordinates this linear map is the Jacobian matrix of ϕ. Given a one-form ω on Q_2, the *pull-back* of ω is the one-form $(\phi^*\omega)$ on Q_1 defined by

$$\langle (\phi^*\omega)(q); X\rangle = \langle \omega; T_q\phi(X)\rangle$$

for all $q \in Q_1$ and $X \in T_qQ_1$.

Now consider a linkage consisting of n rigid bodies with the configuration space Q. Recall that the extended forward kinematics function $\kappa : Q \to SE^n(3)$ from equation (1.5) maps a configuration $q \in Q$ to the configuration of each of the links, i.e., to a vector $(g_{\mathrm{SB}_1}(q), \ldots, g_{\mathrm{SB}_n}(q))$ in the Cartesian space $SE^n(3)$. Forces and velocities in the Cartesian space are described as twists and wrenches; they are thus elements of $se^n(3)$ and $se^{*n}(3)$, respectively. Let W_i be the wrench acting on the ith link. The total force in the Cartesian space is thus $W = (W_1, \ldots, W_n)$. The generalized force Y_i is the component of the total configuration space force Y in the direction of q^i. Formally, $Y_i = \langle Y; \frac{\partial}{\partial q^i}\rangle$, where $\frac{\partial}{\partial q^i}$ is the ith coordinate vector field.

It turns out that the total configuration space force Y is the pull-back of the total Cartesian space force W to T^*Q through the extended forward kinematics map, i.e.,

$$Y = \kappa^*(W).$$

Using this relation and the definition of the pull-back we thus have

$$Y_i(q) = \big\langle Y(q); \frac{\partial}{\partial q^i}\big\rangle = \big\langle W(\kappa(q)); T_q\kappa\big(\frac{\partial}{\partial q^i}\big)\big\rangle. \tag{1.15}$$

It is well established that in the absence of external forces, the generalized force Y_i is the torque applied at the ith joint it the joint is revolute and the force applied at the ith joint if the joint is prismatic; let us formally derive this result using equation (1.15). Consider a serial linkage with the forward kinematics map given by (1.7). Assume that the ith joint connecting $(i-1)$th and ith link is revolute and let τ_i be the torque applied at the joint. At $q \in Q$, we compute

$$T_q\kappa\big(\frac{\partial}{\partial q^j}\big) = (\underbrace{0, \ldots, 0}_{j-1}, \xi_{\mathrm{S},j}(q), \ldots, \xi_{\mathrm{S},j}(q)),$$

where $\xi_{S,j}(q)$ is defined by equation (1.6) and it is the unit twist corresponding to a rotation about the jth joint axis. The wrench in the Cartesian space resulting from the torque τ_i acting along the ith joint axis is $W = \tau_i\,(\underbrace{0,\ldots,0}_{i-2}, -\sigma_{S,i}(q), \sigma_{S,i}(q), 0, \ldots, 0)$, where $\sigma_{S,i}(q)$ is the unit wrench corresponding to a torque about the ith joint axis. It is then easy to check that W results in

$$Y_j = \begin{cases} 0, & j \neq i, \\ \tau_i, & j = i. \end{cases}$$

1.2.4 Geometric interpretation

Additional differential geometric insight into the dynamics can be gained by observing that the kinetic energy provides a *Riemannian metric* on the configuration manifold Q, i.e., a smoothly changing rule for computing the inner product between tangent vectors. Recall that, given $(q, \dot{q}) \in TQ$, the kinetic energy of a manipulator is $T(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q}$, where $M(q)$ is a positive-definite matrix. If we have two vectors $v_1, v_2 \in T_qQ$, we can thus define their inner product by $\langle\langle v^1\,,\, v^2 \rangle\rangle = \frac{1}{2}(v^1)^T M(q) v^2$. Physical properties of a manipulator guarantee that $M(q)$ and thus the rule for computing the inner product changes smoothly over Q. For additional material on this section we refer to [11].

Given two smooth vector fields X and Y on Q, the *covariant derivative* of Y with respect to X is the vector field $\nabla_X Y$ with coordinates

$$(\nabla_X Y)^i = \frac{\partial Y^i}{\partial q^j} X^j + \widehat{\Gamma}^i_{jk} X^j Y^k\,,$$

where X^i and Y^j are the ith and jth component of X and Y, respectively. The operator ∇ is called an *affine connection* and it is determined by the n^3 functions $\widehat{\Gamma}^i_{jk}$. A direct calculation shows that for real valued functions f and g defined over Q, and vector fields X, Y and Z:

$$\begin{aligned} \nabla_{fX+gY} Z &= f\nabla_X Z + g\nabla_Y Z, \\ \nabla_X(Y+Z) &= \nabla_X Y + \nabla_X Z, \\ \nabla_X(fY) &= f\nabla_X Y + X(f)Y, \end{aligned}$$

where in coordinates, $X(f) = \sum_{i=1}^n \frac{\partial f}{\partial q^i} X^i$.

When the functions $\widehat{\Gamma}^i_{jk}$ are computed according to

$$\widehat{\Gamma}^i_{jk}(q) = \frac{1}{2}\sum_{l=1}^n M^{li}\left(\frac{\partial M_{lj}(q)}{\partial q^k} + \frac{\partial M_{lk}(q)}{\partial q^j} - \frac{\partial M_{jk}(q)}{\partial q^l}\right) \tag{1.16}$$

where $M^{li}(q)$ are the components of $M^{-1}(q)$, they are called the *Christoffel symbols (of the second kind)* of the Riemannian metric M, and the affine connection is called the *Levi-Civita connection*. From equations (1.13) and (1.16) it is easy to see that for a Levi-Civita connection,

$$\Gamma_{ijk} = \sum_{l=1}^n M_{li}\widehat{\Gamma}^l_{jk}.$$

Assume that the potential forces are not present so that $L(q, \dot{q}) = T(q, \dot{q})$. The manipulator dynamics equations (1.14) can be thus written as:

$$\ddot{q} + M^{-1}(q)C(q, \dot{q})\dot{q} = M^{-1}(q)Y.$$

It can be seen that the last equation can be written in coordinates as

$$\ddot{q}^i + \widehat{\Gamma}^i_{jk}(q)\dot{q}^j\dot{q}^k = F^i,$$

and in vector format as

$$\nabla_{\dot{q}}\dot{q} = F, \tag{1.17}$$

where ∇ is the Levi-Civita connection corresponding to $M(q)$ and $F = M^{-1}(q)Y$ is the vector field obtained from the one-form Y through the identity $\langle Y; X\rangle = \langle\langle F\,,\, X\rangle\rangle$. In the absence of the external forces, equation (1.17) becomes the so-called *geodesic equation* for the metric M. It is a second order differential equation whose solutions are curves of locally minimal length – like straight lines in $\mathbb{R}^n$ or great circles on a sphere. It can be also shown that among all the curves $\gamma : [0, 1] \to Q$ that connect two given points, a geodesic is the curve that minimizes the energy integral:

$$E(\gamma) = \int_0^1 T(\gamma(t), \dot{\gamma}(t))dt.$$

This geometric insight implies that if a mechanical system moves freely with no external forces present, it will move along the minimum energy paths.

When the potential energy is present, equation (1.17) still applies if the resulting conservative forces are included in F. The one-form Y describing the conservative forces associated with the potential energy V is the differential of V, that is, $Y = -dV$, where the *differential* dV is defined by $\langle dV; X\rangle = X(V)$. The corresponding vector field $F = -M^{-1}(q)dV$ is related to the notion of *gradient* of V, specifically, $F = -\operatorname{grad} V$. In coordinates, differential and gradient of V have components

$$(dV)_i = \frac{\partial V}{\partial q^i}, \quad \text{and} \quad (\operatorname{grad} V)^i = M^{ij}\frac{\partial V}{\partial q^j}.$$

1.3 Constrained systems

One of the advantages of Lagrange's formalism is a systematic procedure to deal with constraints. Generalized coordinates in themselves are a way of dealing with constraints. For example, for robot manipulators, joints limit the relative motion between the links and thus represent constraints. We can avoid modeling these constraints explicitly by choosing joint variables as generalized coordinates. However, in robotics it is often necessary to constrain the motion of the end-effector in some way, model rolling of the wheels of a mobile robot without slippage, or for example take into account various conserved quantities for space robots. Constraints of this nature are external to the robot itself so it is desirable to model them explicitly. Mathematically, a constraint can be described by an equation of the form $\varphi(q, \dot{q}) = 0$, where $q \in \mathbb{R}^n$ are the generalized coordinates and $\varphi : TQ \to \mathbb{R}^m$. However, there is a fundamental difference between constraints of the form $\varphi(q) = 0$, called *holonomic constraints*, and those where the dependence on the generalized velocities can not be eliminated through the integration, known as *nonholonomic constraints*. Also, among the nonholonomic constraints only those that are linear in the generalized velocities turn out to be interesting in practice. In other words, typical nonholonomic constraints are of the form $\varphi(q, \dot{q}) = A(q)\dot{q}$.

Holonomic constraints restrict the motion of the system to a submanifold of the configuration manifold Q. For nonholonomic constraints such a constraint manifold does not exist. However, formally the holonomic and nonholonomic constraints can be treated in the same way by observing that a holonomic constraints $\varphi(q) = 0$ can be differentiated to obtain $A(q)\dot{q} = 0$, where $A(q) = \dfrac{\partial \varphi}{\partial q} \in \mathbb{R}^{m \times n}$. The general form of constraints can be thus assumed to be:

$$A(q)\dot{q} = 0. \tag{1.18}$$

We will assume that $A(q)$ has full rank everywhere.

Constraints are enforced on the mechanical system via a constraint force. A typical assumption is that the constraint force does no work on the system. This excludes examples such as the end-effector sliding with friction along a constraint surface. With this assumption, the constraint force must be at all times perpendicular to the velocity of the system (strictly speaking, it must be in the annihilator of the set of admissible velocities). Since the velocity satisfies equation (1.18), the constraint force, say Λ, will be of the form

$$\Lambda = A^T(q)\lambda, \tag{1.19}$$

where $\lambda \in \mathbb{R}^m$ is a set of *Lagrange multipliers*. The constrained Euler-Lagrange equations thus take the form:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = Y + A^T(q)\lambda. \tag{1.20}$$

These equations, together with the constraints (1.18), determine all the unknown quantities (either Y and λ if the motion of the system is given, or trajectories for q and λ if Y is given).

Several methods can be used to eliminate the Lagrange multipliers and simplify the set of equations (1.18)-(1.20). For example, let $S(q)$ be a matrix whose columns are the basis of the null-space of $A(q)$. Thus we can write:

$$\dot{q} = S(q)\eta \tag{1.21}$$

for some vector $\eta \in \mathbb{R}^{n-m}$. The components of η are called *pseudo-velocities*. Now multiply (1.20) on the left with $S^T(q)$ and use equation (1.21) to eliminate $\dot{q}$. The dynamic equations then become:

$$S^T(q)M(q)S(q)\dot{\eta} + S^T(q)M(q)\frac{\partial S(q)}{\partial q}S(q)\eta + S^T(q)C(q, S(q)\eta) + S^T(q)G(q) = S^T(q)Y. \tag{1.22}$$

The last equation together with equation (1.21) then completely describes the system subject to constraints. Sometimes, such procedures are also known as *embedding of constraints* into the dynamic equations.

1.3.1 Geometric interpretation

The constrained Euler-Lagrange equation can also be given an interesting geometric interpretation. As we did in Section 1.2.4, consider a manifold Q with the Riemannian metric M. Equation (1.18) only allows the system to move in the directions given by the null-space of $A(q)$. Geometrically, the velocity of the system at each point $q \in Q$ must lie in the subset $\mathcal{D}(q) = A(q)$ of the tangent space T_qQ. Formally, $\mathcal{D} = \cup_{q\in Q}\mathcal{D}(q)$ is a *distribution* on Q. For obvious reasons, this distribution will be called the *constraint distribution*. Note that given a constraint distribution $\mathcal{D}$, equation (1.18) simply becomes $\dot{q} \in \mathcal{D}(q)$.

Let $P : TQ \to \mathcal{D}$ denote the orthogonal (with respect to the metric M) projection onto the distribution of feasible velocities $\mathcal{D}$. In other words, at each $q \in Q$, $P(q)$ maps T_qQ into $\mathcal{D}(q) \subset T_qQ$. Let $\mathcal{D}^\perp$ denote the orthogonal complement to $\mathcal{D}$ with respect to the metric M and let $P^\perp = I - P$, where I is the identity map on TQ. Geometrically, the equations (1.18) and (1.20) can be written as:

$$\nabla_{\dot{q}}\dot{q} = \Lambda(t) + F, \tag{1.23}$$

$$P^\perp(\dot{q}) = 0, \tag{1.24}$$

where $\Lambda(t) \in \mathcal{D}^\perp$ is the same as that in equation (1.19) and is the Lagrange multiplier enforcing the constraint.

It is known, e.g., see [9, 17] that equation (1.23) can be written as:

$$\widetilde{\nabla}_{\dot{q}}\dot{q} = P(Y), \tag{1.25}$$

where $\widetilde{\nabla}$ is the affine connection given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \nabla_X\left(P^\perp(Y)\right) - P^\perp(\nabla_X Y). \tag{1.26}$$

We refer to the affine connection $\widetilde{\nabla}$ as the *constraint connection*. Typically, the connection $\widetilde{\nabla}$ is only applied to the vector fields that belong to $\mathcal{D}$. A short computation shows that for $Y \in \mathcal{D}$:

$$\widetilde{\nabla}_X Y = P(\nabla_X Y) \tag{1.27}$$

The last expression allows us to evaluate $\widetilde{\nabla}_X Y$ directly and thus avoid significant amount of computation needed to explicitly compute $\widetilde{\nabla}$ from equation (1.26). Also, by choosing a basis for $\mathcal{D}$, we can directly derive equation (1.22) from (1.25).

The important implication of this result is that even when a system is subject to holonomic or nonholonomic constraints it is indeed possible to write the equations of motion in the form (1.17). In every case the systems evolve over the same manifold Q but they are described with different affine connections. This observation has important implications for control of mechanical systems in general and methods developed for the systems in the form (1.17) can be quite often directly applied to systems with constraints.

1.4 Impact equations

Quite often, robot systems undergo impacts as they move. A typical example are walking robots, but impacts also commonly occur during manipulation and assembly. In order to analyze and effectively control such systems it is necessary to have a systematic procedure for deriving the impact equations. We describe a model for elastic and plastic impacts.

In order to derive the impact equations recall that the Euler-Lagrange equations (1.10) are derived from the *Lagrange-d'Alembert principle*. First, for a given C^2 curve $\gamma : [a,b] \to Q$, define a *variation* $\sigma : (-\epsilon, \epsilon) \times [a,b] \to Q$, a C^2 map with the properties:

(i) $\sigma(0,t) = \gamma(t)$;

(ii) $\sigma(s,a) = \gamma(a)$, $\sigma(s,b) = \gamma(b)$.

Let $\delta q = \frac{d}{ds}\big|_{s=0} \sigma(s,t)$. Note that δq is a vector field along γ. A curve $\gamma(t)$ satisfies the Lagrange-d'Alembert Principle for the force Y and the Lagrangian $L(q,\dot{q})$ if for every variation $\sigma(s,t)$:

$$\frac{d}{ds}\bigg|_{s=0} \int_a^b L\left(\sigma(s,t), \frac{d}{dt}\sigma(s,t)\right)\, dt + \int_a^b \langle Y; \delta q\rangle \, dt = 0. \tag{1.28}$$

Now assume a robot moving on a submanifold $M_1 \subseteq Q$ before the impact, on $M_2 \subseteq Q$ after the impact, with the impact occurring on a $M_3 \subseteq Q$, where $M_3 \subseteq M_1$ and $M_3 \subseteq M_2$. Typically, we would have $M_1 = M_2$ when the system bounces off M_3 (impact with some degree of restitution), or $M_2 = M_3 \subseteq M_1$ for a plastic impact. Assume that $M_i = \varphi_i^{-1}(0)$, where $\varphi_i : Q \to \mathbb{R}^{p_i}$. In other words, we assume that 0 is a regular value of φ_i and that the submanifold M_i corresponds to the zero set of φ_i. The value p_i is the co-dimension of the submanifold M_i. Let τ be the time at which the impact occurs. A direct application of (1.28) leads to the following equation:

$$\int_a^b \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right) \delta q \, dt + \int_a^b \langle Y; \delta q\rangle \, dt + \left(\frac{\partial L}{\partial \dot{q}}\bigg|_{t=\tau^-} - \frac{\partial L}{\partial \dot{q}}\bigg|_{t=\tau^+}\right) \delta q|_{t=\tau} = 0.$$

Given that δq is arbitrary, the first two terms would yield the constrained Euler-Lagrange equations (1.20). The impact equations can be derived from the last term:

$$\left(\frac{\partial L}{\partial \dot{q}}\bigg|_{t=\tau^-} - \frac{\partial L}{\partial \dot{q}}\bigg|_{t=\tau^+}\right) \delta q|_{t=\tau} = 0. \tag{1.29}$$

Since $\gamma(s,\tau) \in M_3$, we have $d\varphi_3 \cdot \delta q|_{t=\tau} = 0$. Equation (1.29) thus implies:

$$\left(\frac{\partial L}{\partial \dot{q}}\bigg|_{t=\tau^-} - \frac{\partial L}{\partial \dot{q}}\bigg|_{t=\tau^+}\right) \in \operatorname{span} d\varphi_3. \tag{1.30}$$

The expression $\frac{\partial L}{\partial \dot{q}}$ can be recognized as the momentum. Equation (1.30) thus leads to the expected geometric interpretation: the momentum of the system during the impact can only change in the directions "perpendicular" to the surface on which the impact occurs.

Let $\varphi_3^1, \ldots, \varphi_3^{p_3}$ be the components of φ_3, and let $\operatorname{grad}\varphi_3^j$ be the gradient of φ_3^j. Observe that $\langle \frac{\partial L(q,\dot{q})}{\partial \dot{q}}; V\rangle = \langle\!\langle \dot{q}\,, V\rangle\!\rangle$. Equation (1.30) can be therefore written as:

$$(\dot{q}|_{t=\tau^-} - \dot{q}|_{t=\tau^+}) = \sum_{j=1}^{p_3} \lambda_{3j} \operatorname{grad}\varphi_3^j. \tag{1.31}$$

In the case of a bounce, the equation suggests that the velocity in the direction orthogonal (with respect to the Riemannian metric M) to the impact surface undergoes the change; the amount of change depends on the coefficient of restitution. In the case of plastic impact, the velocity should be orthogonally (again, with respect to the metric M) projected to the impact surface. In other words, using appropriate geometric tools the impact can indeed be given the interpretation that agrees with the simple setting typically taught in the introductory physics courses.

1.5 Bibliographic remarks

The literature on robot dynamics is vast and the following list is just a small sample. A good starting point to learn about robot kinematics and dynamics are many excellent robotics textbooks [3, 10, 21, 26, 29]. The earliest formulations of the Lagrange equations of motions for robot manipulators are usually considered to be [32] and [15]. Computationally efficient procedures for numerically formulating dynamic equations of serial manipulators using Newton-Euler formalism are described in [19, 22, 30], and in Lagrange formalism in [14]. A comparison between the two approaches is given in [28]. These algorithms have been generalized to more complex structures in [4, 12, 18]. A good review of robot dynamics algorithms is [13]. Differential geometric aspects of modeling and control of mechanical systems are the subject of [1, 6, 9, 20, 27]. Classical dynamics algorithms are recast using differential geometric tools in [31] and [24] for serial linkages, and in [25] for linkages containing closed kinematic chains. Control and passivity of robot manipulators and electromechanical systems are discussed in [2, 23]. Impacts are extensively studied in [8]. An interesting analysis is also presented in [16].

1.6 Conclusion

The chapter gives an overview of the Lagrange formalism for deriving the dynamic equations for robotic systems. The emphasis is on the geometric interpretation of the classical results as such interpretation leads to a very intuitive and compact treatment of constraints and impacts. We start by providing a brief overview of kinematics of serial chains. Rigid-body transformations are described with exponential coordinates; the product of exponentials formula is the basis for deriving forward kinematics map and for the velocity analysis. The chosen formalism allows us to provide a direct mathematical interpretation to many classical notions from screw calculus. Euler-Lagrange equations for serial linkages are presented next. First, they are stated in the traditional coordinate form. The equations are then rewritten in the Riemannian setting using the concept of affine connections. This form of Euler-Lagrange equations enables us to highlight their variational nature. Next, we study systems with constraints. Again, the constrained Euler-Lagrange equations are first presented in their coordinate form. We then show how they can be rewritten using the concept of a constraint connection. The remarkable outcome of this procedure is that the equations describing systems with constraints have exactly the same form as those for unconstrained systems, what changes is the affine connection that is used. We conclude the chapter with a derivation of equations for systems that undergo impacts. Using geometric tools we show that the impact equations have a natural interpretation as a linear velocity projection on the tangent space of the impact manifold.

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