

Metrics and Connections for Rigid Body Kinematics

Miloš Žefran[†], Vijay Kumar[†] and Christopher Croke[‡]

[†]General Robotics and Active Sensory Perception (GRASP) Laboratory

[‡]Department of Mathematics
University of Pennsylvania

Abstract

The set of rigid body motions forms the Lie group $SE(3)$, the special Euclidean group in three dimensions. In this paper we investigate Riemannian metrics and affine connections on $SE(3)$ that are suited for kinematic analysis and robot trajectory planning. In the first part of the paper, we study metrics whose geodesics are screw motions. We prove that no Riemannian metric can have such geodesics and we show that metrics whose geodesics are screw motions form a two-parameter family of semi-Riemannian metrics. In the second part of the paper we investigate affine connections which through the covariant derivative give the expression for the acceleration of a rigid body that agrees with the expression used in kinematics. We prove that there is a unique symmetric connection with this property. Further, we show that there is a family of Riemannian metrics that are compatible with such a connection. These metrics are products of the bi-invariant metric on the group of rotations and a positive definite constant metric on the group of translations.

1 Introduction

The configuration space plays a central role in design, analysis, and control of robot mechanisms. In many applications the focus is on the robot end-effector and its configuration space. Formally, this space consists of all possible displacements of the end-effector and is known as the special Euclidean group, $SE(3)$. This group admits the structure of a differentiable manifold and is therefore a *Lie group* [4, 5]. The tangent space of $SE(3)$ at the identity endowed with the Lie bracket operation has the structure of a Lie algebra and is denoted by $se(3)$. The Lie algebra $se(3)$ is isomorphic to the set of twists [9].

It is customary to consider the configuration space to be a subset of a Euclidean space. However, this often complicates the geometric analysis since the results may depend on the way the configuration space is parameterized (embedded in the Euclidean space). Alternatively, the intrinsic geometric properties of the configuration space can be studied abstractly using differential geometry. In this case, many notions that are well defined in a Euclidean space have to be properly generalized to $SE(3)$.

In this paper we focus on how to differentiate vectors on $SE(3)$ and measure their length. In the language of differential geometry, we study *affine connections* and *Riemannian metrics* on the group $SE(3)$. We are primarily motivated by applications in motion planning, but the results are also applicable to control and design of robots.

A Riemannian metric is everywhere positive definite and it provides a notion of length of curves on the manifold. In contrast, a metric which is non-degenerate but indefinite is called a semi-Riemannian metric [1] and in this case it is more appropriate to speak about the energy of a curve. Curves that minimize the energy between two given points are of particular interest. Such

curves are called *geodesics* and can be considered a generalization of straight lines in Euclidean space \mathbb{R}^n to Riemannian manifolds.

Screw motions have been often used in robotics for trajectory planning [12, 15, 16]. Since Chasles's theorem guarantees the existence of a screw motion between any two points on $SE(3)$, a natural question to ask is whether there exists a metric for which every geodesic is a screw motion. The first important result of this paper is that there is no Riemannian metric with such a property. We further prove that all metrics which have screw motions as geodesics belong to a two-parameter family of semi-Riemannian metrics.

Instead of endowing $SE(3)$ with a metrical structure, we can give it a weaker structure by defining an *affine connection*. An affine connection leads to the definition of the covariant derivative of a vector field along a general curve. In the context of kinematics, the motion of a rigid body is a curve on $SE(3)$ and the velocity at any point is the tangent vector to the curve at that point. We need the definition of a covariant derivative before we can talk about the acceleration of the rigid body. The second key result of the paper concerns a connection that through the covariant derivative of the velocity yields the acceleration used in kinematics¹. We derive the family of connections having this property. Further, we show that among these there is a unique symmetric connection. Finally, we prove that this symmetric connection is compatible with a family of Riemannian metrics that are products of a bi-invariant metric on the group of rotations and a positive definite constant metric on the group of translations.

The paper is organized as follows. In Section 2, we briefly review some basics of differential geometry and Lie groups needed for the rest of the paper. A reader familiar with differential geometry can skip this section. In Section 3, we discuss Chasles's theorem, screw motions and screw displacements. We then prove the first main result of this paper, Theorem 3.5, which identifies metrics for which screw motions are geodesics. In Section 4, we study affine connections that yield the acceleration used in rigid body kinematics. By requiring symmetry of the connection, we show in Proposition 4.1 that such a connection is unique. The second key result of the paper, Theorem 4.7, identifies the family of Riemannian metrics that are compatible with this connection.

2 Kinematics and differential geometry

2.1 The Lie group $SE(3)$

Consider a rigid body moving in free space. Assume any inertial reference frame F fixed in space and a frame M fixed to the body at point O' as shown in Figure 1. At each instance, the configuration (position and orientation) of the rigid body can be described by a homogeneous transformation matrix, A , corresponding to the displacement from frame F to frame M . The set of all such matrices forms the Lie group $SE(3)$, the special Euclidean group in three-dimensions:

$$SE(3) = \left\{ A \mid A = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}, R \in \mathbb{R}^{3 \times 3}, d \in \mathbb{R}^3, R^T R = I, \det(R) = 1 \right\}.$$

On a Lie group, the tangent space at the identity has the structure of a Lie algebra. The Lie algebra of $SE(3)$, denoted by $se(3)$, is given by:

$$se(3) = \left\{ \begin{bmatrix} \Omega & v \\ 0 & 0 \end{bmatrix} \mid \Omega \in \mathbb{R}^{3 \times 3}, v \in \mathbb{R}^3, \Omega^T = -\Omega \right\}. \quad (1)$$

¹This notion will be made precise in Section 4.1

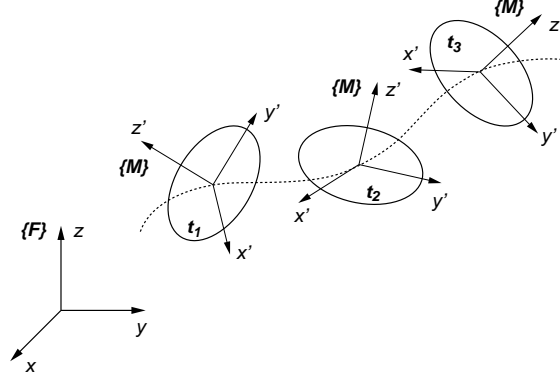


Figure 1: The inertial (fixed) frame and the moving frame attached to the rigid body

A 3×3 skew-symmetric matrix Ω can be uniquely identified with a vector $\omega \in \mathbb{R}^3$ [9]. Each element $S \in se(3)$ can be thus identified with a vector pair $\{\omega, v\}$.

Given a curve $A(t) : [-a, a] \rightarrow SE(3)$, an element $S(t)$ of the Lie algebra $se(3)$ can be associated to the tangent vector $\dot{A}(t)$ at an arbitrary point t by:

$$S(t) = A^{-1}(t)\dot{A}(t). \quad (2)$$

A curve on $SE(3)$ physically represents a motion of the rigid body. If $\{\omega(t), v(t)\}$ is the vector pair corresponding to $S(t)$, then ω physically corresponds to the angular velocity of the rigid body while v is the linear velocity of the origin O' of the frame M , both expressed in the frame M . In kinematics, elements of this form are called twists [8] and $se(3)$ is thus isomorphic to the space of twists.

Since $se(3)$ is a vector space, we can choose a basis and identify any element with a 6×1 vector. The standard basis for $se(3)$ is:

$$L_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad L_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad L_3 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$L_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad L_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad L_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The twists L_1 , L_2 and L_3 represent instantaneous rotations about and L_4 , L_5 and L_6 instantaneous translations along the Cartesian axes x , y and z , respectively. In this basis, the vector corresponding to a twist $S \in se(3)$ is given precisely by the velocity pair $\{\omega, v\}$.

Given two elements $S_1, S_2 \in se(3)$, we can define another element $[S_1, S_2] \in se(3)$ called the *Lie Bracket* by:

$$[S_1, S_2] = S_1 S_2 - S_2 S_1.$$

Since the Lie bracket belongs to $se(3)$, it can be expressed as a linear combination of the basis vectors. The coefficients C_{ij}^k corresponding to the Lie brackets of the basis vectors are called *structure constants* of the Lie algebra [13]:

$$[L_i, L_j] = \sum_k C_{ij}^k L_k. \quad (3)$$

For $se(3)$, the nonzero structure constants are:

$$\begin{aligned} C_{12}^3 = C_{31}^2 = C_{23}^1 = C_{15}^6 = C_{26}^4 = C_{34}^5 = C_{42}^6 = C_{53}^4 = C_{61}^5 = 1 \\ C_{21}^3 = C_{13}^2 = C_{32}^1 = C_{51}^6 = C_{62}^4 = C_{43}^5 = C_{24}^6 = C_{35}^4 = C_{16}^5 = -1 \end{aligned}$$

2.2 Left invariant vector fields and exponential mapping

A *differentiable vector field* is a smooth assignment of a tangent vector to each element of the manifold. An example of a differentiable vector field, X , on $SE(3)$ is obtained by left translation of an element $S \in se(3)$. The value of this vector field at an arbitrary point $A \in SE(3)$ is given by:

$$X(A) = AS, \quad (4)$$

and the vector field is called a *left invariant* vector field. To denote that a left invariant vector field was obtained by left translating the Lie algebra element S , we will also use the notation \hat{S} . By construction, the space of left invariant vector fields is isomorphic to the Lie algebra $se(3)$ [2]. In particular:

$$[\hat{L}_i, \hat{L}_j] = [\widehat{L_i}, \widehat{L_j}] = \sum_k C_{ij}^k \hat{L}_k. \quad (5)$$

Since the vectors L_1, L_2, \dots, L_6 are a basis for the Lie algebra $se(3)$, the vectors $\hat{L}_1(A), \dots, \hat{L}_6(A)$ form a basis of the tangent space at any point $A \in SE(3)$. Therefore, any vector field X can be expressed as

$$X = \sum_{i=1}^6 X^i \hat{L}_i, \quad (6)$$

where the coefficients X^i vary over the manifold. If the coefficients are constants, then X is left invariant. By defining:

$$\omega = [X^1, X^2, X^3]^T, \quad v = [X^4, X^5, X^6]^T,$$

we can associate a vector pair of functions $\{\omega, v\}$ to an arbitrary vector field X . If a curve $A(t)$ describes a motion of the rigid body and $V = \frac{dA}{dt}$ is the vector field tangent to $A(t)$, the vector pair $\{\omega, v\}$ associated with V corresponds to the instantaneous twist (screw axis) for the motion. In general, the twist $\{\omega, v\}$ changes with time. Motions for which the twist $\{\omega, v\}$ is constant are known in kinematics as *screw motions*. If the vector pair $\{\omega, v\}$ is interpreted as Plücker coordinates of a line in space, it is not difficult to see that the screw motion physically corresponds to rotation of the rigid body around this line with a constant angular velocity and concurrent translation of the body along the line with a constant translational velocity.

On $SE(3)$, one can define the exponential map $\exp : se(3) \rightarrow SE(3)$ using the standard matrix exponential [9]. This map relates the screw motion $A(t)$ and the corresponding twist S :

$$\exp(tS) = A(t). \quad (7)$$

2.3 Riemannian metrics on Lie groups

If a smoothly varying, positive definite, bilinear, symmetric form $\langle \cdot, \cdot \rangle$ is defined on the tangent space at each point on the manifold, such a form is called a Riemannian metric and the manifold is Riemannian² [2]. If the form is non-degenerate but indefinite, it is called a

²Under very mild assumptions a differentiable manifold always admits a Riemannian metric.

semi-Riemannian metric[1]. On a n dimensional manifold, the metric is locally characterized by a $n \times n$ matrix of C^∞ functions $g_{ij} = \langle X_i, X_j \rangle$ where X_i are basis vector fields.

On a Lie group, an inner product on the Lie algebra (given in a chosen basis by a positive definite symmetric matrix W) can be extended to a Riemannian metric over the manifold using left (or right) translation. If V_1 and V_2 are tangent vectors at an arbitrary group element $A \in SE(3)$, the inner product $\langle V_1, V_2 \rangle|_A$ in the tangent space $T_A SE(3)$ can be defined by:

$$\langle V_1, V_2 \rangle|_A = \langle A^{-1}V_1, A^{-1}V_2 \rangle|_I. \quad (8)$$

The metric obtained in such a way is said to be left invariant [2] since left translation by any element A is an isometry.

2.4 Affine connection and covariant derivative

The motion of a rigid body can be represented by a curve on $SE(3)$. The velocity at an arbitrary point is the tangent vector to the curve at that point. In order to obtain the acceleration we need to be able to differentiate a vector field along the curve. A derivative of a vector field along a curve $A(t)$ is defined using the concept of *parallel transport*. Let X be a vector field defined along $A(t)$, and let $X(t)$ stand for $X(A(t))$. Denote by $X^{t_0}(t)$ the parallel transport of the vector $X(t)$ to the point $A(t_0)$. The *covariant derivative* of X along $A(t)$ is:

$$\frac{DX}{dt} \Big|_{t_0} = \lim_{t \rightarrow t_0} \frac{X^{t_0}(t) - X(t_0)}{t}. \quad (9)$$

By taking covariant derivatives along integral curves of a vector field Y , we obtain a covariant derivative of the vector field X with respect to the vector field Y . This derivative is also denoted by $\nabla_Y X$:

$$\nabla_Y X|_{A_0} = \frac{DX}{dt} \Big|_{t_0}, \quad (10)$$

and the operator ∇ is called the *affine connection*.

The covariant derivative of a vector field is another vector field so it can be expressed as a linear combination of the basis vector fields. The coefficients Γ_{ji}^k of the covariant derivative of a basis vector field along another basis vector field,

$$\nabla_{\hat{L}_i} \hat{L}_j = \sum_k \Gamma_{ji}^k \hat{L}_k, \quad (11)$$

are called *Christoffel symbols*³. Note the reversed order of the indices i and j .

The velocity, $V(t)$, of the rigid body describing the motion $A(t)$ is given by the tangent vector field along the curve:

$$V(t) = \frac{dA(t)}{dt}.$$

The acceleration, $\mathcal{A}(t)$, is the covariant derivative of the velocity along the curve

$$\mathcal{A} = \frac{D}{dt} \left(\frac{dA}{dt} \right) = \nabla_V V. \quad (12)$$

Note that the acceleration depends on the choice of the connection.

³Different definitions for the Christoffel symbols can be found in the literature. Some texts (e.g. [2]) reserve the term for the case of the coordinate basis vectors. We follow the more general definition from [14] in which the basis vectors can be arbitrary.

Given a Riemannian manifold, there exists a unique connection [2] which is compatible with the metric:

$$X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle^4 \quad (13)$$

and symmetric:

$$\nabla_X Y - \nabla_Y X = [X, Y]. \quad (14)$$

This connection is called the *Levi-Civita* or *Riemannian connection*.

2.5 Geodesics

Given a Riemannian metric $\langle \cdot, \cdot \rangle$ on $SE(3)$ we can define the length, $L(A)$, of a smooth curve $A : [a, b] \rightarrow SE(3)$ by:

$$L(A) = \int_a^b \left\langle \frac{dA}{dt}, \frac{dA}{dt} \right\rangle^{\frac{1}{2}} dt \quad (15)$$

Among all the curves connecting two points, we are usually interested in the curve of minimal length. It is not difficult to see [2] that a curve of minimal length also minimizes the *energy functional*:

$$E(A) = \int_a^b \left\langle \frac{dA}{dt}, \frac{dA}{dt} \right\rangle dt. \quad (16)$$

If a curve minimizes a functional, it must be also a critical point. Critical points of the energy functional E satisfy the following equation [2]:

$$\nabla_{\frac{dA}{dt}} \frac{dA}{dt} = 0, \quad (17)$$

where ∇ is the Riemannian connection, and are called *geodesics*. According to Equation (12), the expression $\nabla_{\frac{dA}{dt}} \frac{dA}{dt}$ is the acceleration of motion described by $A(t)$. Motion along a geodesic therefore produces zero acceleration.

3 Metrics and screw motions

One of the fundamental results in rigid body kinematics [8] was proved by Chasles at the beginning of the 19th century:

Theorem 3.1 (Chasles) *Any rigid body displacement can be realized by a rotation about an axis combined with a translation parallel to that axis.*

Note that a displacement must be understood as an element of $SE(3)$ while a motion is a curve on $SE(3)$. In Chasles's theorem, the rotation can be performed at constant angular velocity and the translation at constant translational velocity. Hence, we conclude that any rigid body displacement can be realized through a screw motion.

Another family of curves of interest on a Lie group is the set of the *one-parameter subgroups*. A curve $A(t)$ is a one-parameter subgroup, if $A(t_1 + t_2) = A(t_1)A(t_2)$. The one-parameter subgroups on $SE(3)$ are given by [2]:

$$A(t) = \exp(tS) \quad (18)$$

⁴Note that formally, a vector field X is a derivation operator which, given a differentiable function, returns its derivative along the integral curves of X . The expression $X \langle Y, Z \rangle$ is therefore a derivative of the function $\langle Y, Z \rangle$ along the integral curves of X .

where S is an element of $se(3)$. Comparing with Equation (7), we conclude that the one parameter subgroups are exactly the screw motions which pass through the identity. The following proposition follows from the Lie group structure of $SE(3)$ and is directly related to Theorem 3.1:

Proposition 3.2 *If A_1 and A_2 are two distinct elements of $SE(3)$, then:*

- (1) *There exists a one-parameter subgroup, $\gamma_L(t) = \exp(tS_L)$, which when left translated by A_1 contains A_2 :*

$$A_L(t) = A_1 \exp(tS_L), \quad A_2 = A_L(1) = A_1 \exp(S_L).$$

- (2) *There exists a one-parameter subgroup, $\gamma_R(t) = \exp(tS_R)$, which when right translated by A_1 contains A_2 :*

$$A_R(t) = \exp(tS_R)A_1, \quad A_2 = A_R(1) = \exp(S_R)A_1.$$

- (3) *$A_L(t)$ and $A_R(t)$ are the same screw motion and S_L in (1) is related to S_R in (2) by:*

$$S_R = A_1 S_L A_1^{-1}.$$

3.1 Screw motions as geodesics

Given that any two elements of $SE(3)$ can be connected with a screw motion, it is natural to ask whether screw motions are geodesics (minimum energy curves) for any metric. In [11], Park and Brockett proposed a left invariant Riemannian metric on $SE(3)$ given by:

$$W = \begin{bmatrix} \alpha I & 0 \\ 0 & \beta I \end{bmatrix} \tag{19}$$

where α and β are positive scalars. Park [10] showed that the geodesics for this metric are products of the geodesics for the bi-invariant metric on $SO(3)$ and geodesics in the Euclidean space \mathbb{R}^3 . It is not difficult to show that a screw motion is a geodesic for the metric (19) if and only if it is obtained by a left translation of a one-parameter subgroup for which the screw axis passes through the origin of the inertial frame F . The question now becomes whether there is a metric for which every screw motion is a geodesic.

Suppose that the screw motions are geodesics. Since a left or a right translation of a screw motion produces another screw motion (Proposition 3.2), we might be led to conclude that any metric for which the screw motions are geodesics must be invariant under left and right translations and therefore bi-invariant. Such reasoning is flawed since a map which preserves geodesics does not necessarily preserve the metric (is not necessarily an isometry). For example, affine transformations in \mathbb{R}^3 map lines into lines (that is, they map geodesics into geodesics), but in general, they do not preserve lengths of vectors. Diffeomorphisms that take geodesics to geodesics (but make an affine change of parameter) are called *affine maps*. They are rare, except for isometries, affine maps on Euclidean spaces, and products of such. They have been studied in [7, Chapter IV]. We will not limit our search to left or right invariant metrics, nor will we need the results of [7].

Since the twist associated with a screw motion $\gamma(t)$ is constant, the tangent vector field $V = \frac{d\gamma}{dt} = V^i \hat{L}_i$ is a left invariant vector field and the components V^i are constant. If γ solves Equation (17), we have:

$$0 = \nabla_V V = \sum_j \frac{dV^j}{dt} \hat{L}_j + \sum_{i,j} V^i V^j \nabla_{\hat{L}_i} \hat{L}_j = \sum_{i,j} V^i V^j \nabla_{\hat{L}_i} \hat{L}_j. \quad (20)$$

The above equation is satisfied for an arbitrary screw motion (arbitrary choice of the components V^i) if and only if:

$$\begin{aligned} \nabla_{\hat{L}_i} \hat{L}_i &= 0 & 1 \leq i \leq 6, \\ \nabla_{\hat{L}_i} \hat{L}_j + \nabla_{\hat{L}_j} \hat{L}_i &= 0 & 1 \leq i < j \leq 6. \end{aligned}$$

Since ∇ is a metrical connection, it is symmetric (Equation 14):

$$\nabla_{\hat{L}_i} \hat{L}_j - \nabla_{\hat{L}_j} \hat{L}_i = [\hat{L}_i, \hat{L}_j].$$

It immediately follows from the above equations that:

$$\nabla_{\hat{L}_i} \hat{L}_j = \frac{1}{2} [\hat{L}_i, \hat{L}_j]. \quad (21)$$

(Note that $[\hat{L}_i, \hat{L}_i] = 0$.) Further, ∇ must be compatible with the metric (Equation 13), so we have:

$$\hat{L}_k \langle \hat{L}_i, \hat{L}_j \rangle = \langle \nabla_{\hat{L}_k} \hat{L}_i, \hat{L}_j \rangle + \langle \hat{L}_i, \nabla_{\hat{L}_k} \hat{L}_j \rangle. \quad (22)$$

Letting $g_{ij} = \langle \hat{L}_i, \hat{L}_j \rangle$, the last equation implies:

$$\hat{L}_k(g_{ij}) = \frac{1}{2} \left(\langle [\hat{L}_k, \hat{L}_i], \hat{L}_j \rangle + \langle \hat{L}_i, [\hat{L}_k, \hat{L}_j] \rangle \right). \quad (23)$$

By expressing the Lie brackets from Equation (3), we finally obtain:

$$\hat{L}_k(g_{ij}) = \frac{1}{2} \sum_l (C_{ki}^l g_{lj} + C_{kj}^l g_{li}). \quad (24)$$

Note that the coefficients C_{ij}^k are constant over the manifold (Equation 5). The above derivation can be summarized in the following proposition:

Proposition 3.3 *Screw motions satisfy the geodesic equation (17) for a Riemannian metric given by the matrix of coefficients $G = [g_{ij}]$ if and only if the coefficients g_{ij} satisfy Equation (24).*

The metric coefficients g_{ij} are symmetric by definition. Since $SE(3)$ is a 6 dimensional manifold, there are 21 independent coefficients $\{g_{ij} \mid 1 \leq i \leq j \leq 6\}$. Given that there are 6 basis vector fields, Equation (24) expands to a total of 126 partial differential equations. The complete set of equations is given in Equation (58) in Appendix A.

The following lemma will be useful in deriving the solution for the system of equations (24):

Lemma 3.4 *Given a set of partial differential equations*

$$X(f) = g_1 \quad (25)$$

$$Y(f) = g_2 \quad (26)$$

$$Z(f) = g_3 \quad (27)$$

where X , Y , and Z are vector fields such that $Z = [X, Y]$, f is twice differentiable, and g_1 , g_2 and g_3 are differentiable (real valued) functions, the solution exists only if

$$X(g_2) - Y(g_1) = g_3. \quad (28)$$

Proof: By applying X on Equation (26), Y on Equation (25) and subtracting the two resulting equations, we get:

$$X Y(f) - Y X(f) = X(g_2) - Y(g_1). \quad (29)$$

But the left-hand side is by definition $[X, Y](f)$, which is by assumption equal to $Z(f)$. Equation (28) then follows from Equation (27). \square

We next state the first key theorem of this section:

Theorem 3.5 *A matrix of coefficients $G = [g_{ij}]$ satisfies the system of partial differential equations (24) if and only if it has the form*

$$G = \begin{bmatrix} \alpha I_{3 \times 3} & \beta I_{3 \times 3} \\ \beta I_{3 \times 3} & 0_{3 \times 3} \end{bmatrix}, \quad (30)$$

where α and β are constants.

Proof: To find the metric coefficients, we start with the following subset of (58):

$$\widehat{L}_1(g_{11}) = 0 \quad \widehat{L}_2(g_{11}) = -g_{13} \quad \widehat{L}_3(g_{11}) = g_{12}. \quad (31)$$

First, we observe that $[\widehat{L}_1, \widehat{L}_2] = \widehat{L}_3$. By application of Lemma 3.4, the following equation must hold:

$$-\widehat{L}_1(g_{13}) = g_{12}. \quad (32)$$

But from (58) we also have:

$$\widehat{L}_1(g_{13}) = -\frac{1}{2}g_{12},$$

which implies that $g_{12} = 0$. We next observe that $g_{12} = 0$ implies $\widehat{L}_i(g_{12}) = 0$, $1 \leq i \leq 6$. From the system (58) we obtain:

$$\begin{aligned} g_{13} = 0 & \quad g_{23} = 0 & \quad g_{11} = g_{22} \\ g_{16} = 0 & \quad g_{26} = 0 & \quad g_{14} = g_{25}. \end{aligned} \quad (33)$$

From these equations and (58) we further obtain:

$$\begin{aligned} g_{15} = 0 & \quad g_{24} = 0 & \quad g_{11} = g_{33} \\ g_{34} = 0 & \quad g_{35} = 0 & \quad g_{14} = g_{36} \\ g_{44} = 0 & \quad g_{45} = 0 & \quad g_{46} = 0 \\ g_{55} = 0 & \quad g_{56} = 0 & \quad g_{66} = 0. \end{aligned} \quad (34)$$

Next observation is that $\widehat{L}_i(g_{11}) = 0$, $1 \leq i \leq 6$. This, together with Equations (33) and (34) implies:

$$g_{11} = g_{22} = g_{33} = \alpha,$$

where α is an arbitrary constant. Similarly, we obtain

$$g_{14} = g_{25} = g_{36} = \beta,$$

for an arbitrary constant β . In this way we have obtained all 21 independent values of G . The reader can easily check that the above values satisfy the system of equations (58). \square

Corollary 3.6 *There is no Riemannian metric whose geodesics are screw motions.*

Proof: A matrix of the form

$$G = \begin{bmatrix} \alpha I_{3 \times 3} & \beta I_{3 \times 3} \\ \beta I_{3 \times 3} & 0_{3 \times 3} \end{bmatrix},$$

has two distinct real eigenvalues

$$\lambda_1 = \frac{1}{2}(\alpha + \sqrt{\alpha^2 + 4\beta^2}) \quad \lambda_2 = \frac{1}{2}(\alpha - \sqrt{\alpha^2 + 4\beta^2}),$$

which both have multiplicity 3. For any choice of α and β , the product of the eigenvalues is $\lambda_1 \lambda_2 = -\beta^2 \leq 0$. Therefore, G is not positive definite as required for a Riemannian metric. \square

3.2 Geodesics of the family of metrics (30)

Analogous to the Riemannian case, we could define the length of a curve $A(t)$ between two points $A(t_1)$ and $A(t_2)$ on $SE(3)$ by:

$$L(A; t_1, t_2) = \int_{t_1}^{t_2} \left\langle \frac{dA}{dt}, \frac{dA}{dt} \right\rangle^{\frac{1}{2}} dt. \quad (35)$$

But members of the family (30) are not positive definite, so the length of a curve would be in general a complex number. It is therefore more useful to define the measure of the energy of a curve:

$$E(A; t_1, t_2) = \int_{t_1}^{t_2} \left\langle \frac{dA}{dt}, \frac{dA}{dt} \right\rangle dt. \quad (36)$$

For a metric in the family (30) the energy of a curve can be in general negative. There are also non-trivial curves (that is, curves that are not identically equal to a point) which have zero energy.

Two special cases of metric (30) are of particular interest. With $\alpha = 0$ and $\beta = 1$ we obtain the metric:

$$G = \begin{bmatrix} 0_{3 \times 3} & I_{3 \times 3} \\ I_{3 \times 3} & 0_{3 \times 3} \end{bmatrix}.$$

This metric, taken as a quadratic form on $se(3)$, is known as the Klein form [5]. The eigenvalues for the metric are $\{1, 1, 1, -1, -1, -1\}$ and the form is therefore non-degenerate. For a screw motion $A(t) = A_0 \exp(tS)$ where $S = \{\omega, v\} \in se(3)$, the energy of the segment $t \in [0, 1]$ is given by $E(A) = 2 \omega^T v$. If $\omega \neq 0$, the quantity:

$$h = \frac{\omega^T v}{|\omega|^2} \quad (37)$$

is called the *pitch* of the screw motion [6]. The pitch measures the amount of translation along the screw axis during the screw motion. Zero energy screw motions therefore either have zero pitch (the motion is pure rotation) or infinite pitch ($\omega = 0$, the motion is pure translation). Screw motions with positive energy are those with positive pitch. Trajectories for such motions correspond to right-handed helices and the motions are thus called right-handed screw motions. Similarly, screw motions with negative energy are left-handed screw motions. Since pure rotations and pure translations are zero-energy motions, it is always possible to find a zero energy curve between two arbitrary points by breaking the motion into a segment consisting of a pure rotation followed by a segment of a pure translation.

By letting $\alpha = 1$ and $\beta = 0$, we get the semi-definite metric:

$$G = \begin{bmatrix} I_{3 \times 3} & 0_{3 \times 3} \\ 0_{3 \times 3} & 0_{3 \times 3} \end{bmatrix}.$$

This metric, as a form on $se(3)$, is called the Killing form [5]. Its eigenvalues are $\{1, 1, 1, 0, 0, 0\}$ hence it is degenerate. The energy of a screw motion with $S = \{\omega, v\}$ is equal to $\omega^T \omega$ so it is always non-negative. Pure translations are zero-energy motions while any motion involving rotation has positive energy.

In the general case, $\alpha \neq 0$ and $\beta \neq 0$, the energy of a screw motion $A(t) = A_0 \exp(tS)$ where $S = \{\omega, v\}$ and $t \in [0, 1]$, is $\omega^T(\alpha\omega + 2\beta v)$. Pure translations ($\omega = 0$) thus have zero energy. For a general screw motion ($\omega \neq 0$), the energy of the segment $t \in [0, 1]$ is $|\omega|^2(\alpha + 2\beta h)$. The sign of the energy of a general motion therefore depends on α and β .

Remark 3.7 The Klein form and the Killing form on $SE(3)$ are known to be bi-invariant [5]. Since every metric in the family (30) is a linear combination (with constant coefficients) of these two forms, it is clear that all these metrics are bi-invariant.

4 Affine connections on $SE(3)$

There is no natural choice of a metric on $SE(3)$. In the previous section, we chose a particular family of curves (screw motions) and found the metric G for which these curves were geodesics. Through the metric connection, a metric also provides a natural way to differentiate vector fields.

In this section, our primary objective is to find a connection which produces an acceleration vector that is physically meaningful. We therefore start by introducing a connection on $SE(3)$ which allows us to compute the covariant derivative and obtain the acceleration. By requiring that the acceleration computed with the covariant derivative agrees with the acceleration as defined in kinematics using principles of vector calculus (see below), we obtain a family of possible affine connections. We then show that there is a unique symmetric connection in this family. Finally, we determine the class of Riemannian metrics which are compatible with this symmetric affine connection.

4.1 Kinematic connection

Let $A(t)$ be a curve describing a motion of a rigid body. The velocity vector field, $V(t)$, is given by the tangent vector at each point on the curve, $V(t) = \frac{dA}{dt}$. We can write $V(t) = V^i \hat{L}_i$ and define $\{\omega(t), v(t)\}$ to be the corresponding vector of components. The velocity pair $S(t) = \{\omega(t), v(t)\}$ is the instantaneous twist associated with the motion, expressed in the moving frame M fixed to the rigid body. More precisely, ω represents the angular velocity of the rigid body while v is the linear velocity of the origin O' of the body fixed reference frame M (see Figure 1). It is important to note the difference between the twist $S(t)$ used to obtain Equation (38) and the velocity vector field $V(t) = V(A(t))$: the twist $S(t)$ belongs to $se(3)$, while $V(t)$ belongs to the tangent space $T_{A(t)} SE(3)$. We now turn our attention to the acceleration vector. In kinematics, the acceleration is represented by a vector pair $\{\alpha, a\}$, where α is the angular acceleration of the rigid body and a is the acceleration of the point O' , both expressed in the frame M . This acceleration vector pair can be written in the form:

$$\{\alpha, a\} = \{\dot{\omega}, \dot{v}\} + \{0, \omega \times v\}. \quad (38)$$

The first term is simply the derivative of the components of the angular velocity and the linear velocity of the point O' expressed in the frame M . This term is also called the spatial acceleration [3]⁵. However, angular and linear velocities are expressed in the body fixed frame M , which rotates as the body moves. We must therefore add the second term, a convective term that describes the contribution of this rotation to the acceleration of the rigid body. This term is an artifact of expressing the velocities in a frame that rotates.

On the other hand, geometrically, the acceleration, \mathcal{A} , of the rigid body is given by the covariant derivative of the velocity vector field $V = \frac{dA}{dt}$ along $A(t)$:

$$\mathcal{A} = \nabla_V V. \quad (39)$$

According to Equation (38), the affine connection that produces a physically meaningful acceleration must satisfy:

$$\nabla_V V = \{\dot{\omega}, \dot{v}\} + \{0, \omega \times v\}. \quad (40)$$

In components, $\nabla_V V$ can be rewritten as:

$$\nabla_V V = \sum_k \frac{dV^k}{dt} \hat{L}_k + \sum_k \sum_{1 \leq i, j \leq 6} V^i V^j \Gamma_{ji}^k \hat{L}_k, \quad (41)$$

where Γ_{ji}^k are the Christoffel symbols (which define the affine connection) for the basis \hat{L}_i :

$$\nabla_{\hat{L}_i} \hat{L}_j = \sum_k \Gamma_{ji}^k \hat{L}_k. \quad (42)$$

Comparing the right hand side of (40) and (41), it is clear that the first terms are the same regardless of the choice of the affine connection. In both, (40) and (41), the second term on the right hand side depends on the products of velocity components. Recalling that the first three components of V correspond to ω and the last three components correspond to v , we can write the second term of (40) in the form:

$$\{0, \omega \times v\} = \sum_k \sum_{i \leq j} a_{ij}^k V^i V^j \hat{L}_k, \quad (43)$$

where a_{ij}^k are constants that are all zero except for:

$$\begin{aligned} a_{24}^6 &= -1 & a_{34}^5 &= 1 & a_{15}^6 &= 1 \\ a_{35}^4 &= -1 & a_{16}^5 &= -1 & a_{26}^4 &= 1. \end{aligned} \quad (44)$$

Equating the right hand sides of (40) and (41) gives the following relationships:

$$\begin{aligned} \Gamma_{ii}^k &= a_{ii}^k & 1 \leq i \leq 6 \\ \Gamma_{ij}^k + \Gamma_{ji}^k &= a_{ij}^k & 1 \leq i < j \leq 6. \end{aligned} \quad (45)$$

The system (45) does not contain enough equations to solve for Γ_{ij}^k if $i \neq j$ (but we get $\Gamma_{ii}^k = 0$, $1 \leq i, k \leq 6$). Therefore, there is a family of affine connections on $SE(3)$ that produce the acceleration used in kinematics. acceleration.

⁵Featherstone [3] considers the acceleration in a fixed (inertial) frame and demonstrates that it has the form shown in (38). While the ‘‘convective term’’ is present in his derivation, the physical interpretation is different.

To further restrict the choice of the connection, we might ask if any of the connections that satisfy Equation (45) are symmetric. In other words, we require that the connection satisfies Equation (14):

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

It immediately follows that for the basis \widehat{L}_i , the symmetry of the connection implies:

$$\Gamma_{ji}^k - \Gamma_{ij}^k = C_{ij}^k. \quad (46)$$

where C_{ij}^k are the structure constants for $se(3)$ (Equation 4). Equations (45) and (46) together uniquely determine the Christoffel symbols Γ_{ji}^k and therefore the connection. We call this symmetric affine connection the *kinematic connection*. The non-zero Christoffel symbols for the kinematic connection can be easily seen to be:

$$\begin{aligned} \Gamma_{21}^3 = \Gamma_{13}^2 = \Gamma_{32}^1 = \frac{1}{2}, \quad \Gamma_{12}^3 = \Gamma_{31}^2 = \Gamma_{23}^1 = -\frac{1}{2} \\ \Gamma_{51}^6 = \Gamma_{62}^4 = \Gamma_{43}^5 = 1, \quad \Gamma_{42}^6 = \Gamma_{53}^4 = \Gamma_{61}^5 = -1 \end{aligned} \quad (47)$$

We can therefore state the following proposition to summarize the derivations in this section.

Proposition 4.1 *On $SE(3)$, there exists a unique symmetric affine connection, called **the kinematic connection**, that produces the acceleration as defined in kinematics. The Christoffel symbols for the kinematic connection with respect to the basis vector fields \widehat{L}_i are given in Equation (47).*

4.2 Metrics compatible with the kinematic connection

In Section 2.4 we defined a connection to be Riemannian if it is symmetric and compatible with the metric. Since we explicitly required that the kinematic connection be symmetric, we can try to find a metric which is compatible with the connection. In general, such a metric may not exist.

If a metric is compatible with the connection, then:

$$Z \langle X, Y \rangle = \langle \nabla_Z X, Y \rangle + \langle X, \nabla_Z Y \rangle, \quad (48)$$

where X, Y and Z are arbitrary vector fields. By substituting the basis vector fields $\widehat{L}_i, \widehat{L}_j$ and \widehat{L}_k for X, Y and Z , the compatibility condition becomes:

$$\widehat{L}_k(g_{ij}) = \sum_l (\Gamma_{ik}^l g_{lj} + \Gamma_{jk}^l g_{li}) \quad (49)$$

where the Christoffel symbols Γ_{ji}^k were computed above. Equation (49) generates a system of 126 partial differential equations for the 21 metric coefficients $\{g_{ij} \mid 1 \leq i \leq j \leq 6\}$. Note that for $k > 3$, $\Gamma_{ik}^j = 0$, so $\widehat{L}_k(g_{ij}) = 0$. The system of equations obtained from Equation (49) for $k \leq 3$ is given in Equation (59) in Appendix B. The first step in finding the solution of this system of equations is proving the following lemma.

Lemma 4.2 *If the coefficients of a Riemannian metric G satisfy Equation (49), the metric has the form:*

$$G = \begin{bmatrix} \alpha I & 0 \\ 0 & G_p \end{bmatrix}, \quad (50)$$

where α is a constant and G_p is an arbitrary positive definite symmetric matrix that smoothly varies from point to point.

Proof: We use Lemma 3.4 again. Take the following subset of equations of the system (59):

$$\widehat{L}_1(g_{11}) = 0 \quad \widehat{L}_2(g_{11}) = -g_{13} \quad \widehat{L}_3(g_{11}) = g_{12} \quad (51)$$

According to Lemma 3.4 the following equality holds:

$$-\widehat{L}_1(g_{13}) = g_{12}.$$

By substituting for $\widehat{L}_1(g_{13})$ from (59), we obtain $\frac{1}{2}g_{12} = g_{12}$, which gives $g_{12} = 0$. Substituting in the system (59), we next obtain:

$$g_{13} = 0 \quad g_{23} = 0 \quad g_{11} = g_{22} \quad g_{11} = g_{33} \quad (52)$$

It is easy to see that these equations imply $\widehat{L}_i(g_{11}) = 0$, $1 \leq i \leq 6$, which together with Equation (52) results in:

$$g_{11} = g_{22} = g_{33} = \alpha,$$

where α is a constant. Therefore, the upper-left 3×3 block in the matrix G is of the form $\alpha I_{3 \times 3}$, where I is the identity matrix.

By taking equations:

$$\widehat{L}_1(g_{14}) = 0 \quad \widehat{L}_2(g_{14}) = -\frac{1}{2}g_{34} - g_{16} \quad \widehat{L}_3(g_{14}) = \frac{1}{2}g_{24} + g_{15}, \quad (53)$$

and again using Lemma 3.4, we get $g_{24} = 0$. By substituting this in the system (59) it is easy to see that all the entries in the upper-right 3×3 block of the matrix G (and due to the symmetry of G also in the lower-left 3×3 block) are equal to 0, proving that G must be of the form (50). Since we are interested in Riemannian metrics, the matrix G must be positive definite and symmetric. The form of G then implies that also G_p must be positive definite and symmetric. \square

Proposition 4.3 *A left invariant metric is compatible with the kinematic connection if and only if the matrix of metric coefficients $G = [g_{ij}]$ is of the form:*

$$G = \begin{bmatrix} \alpha I & 0 \\ 0 & \beta I \end{bmatrix}, \quad (54)$$

where α and β are arbitrary constants.

Proof: If a metric is left invariant then the matrix G is constant. But if $g_{ij} = \text{const.}$, then $\widehat{L}_i(g_{ij}) = 0$. It is then easy to check that the system of equations (59) implies the form of G in (54). \square

Remark 4.4 Note that the metric (54) is exactly the same as the metric (19).

To determine whether there are other solutions to the system (59), it helps to investigate what is common to metrics that have the same metrical connection. The geometric entity determined from the connection are geodesics (Equation 17). Two metrics with the same connection thus have the same geodesics. By reasoning similar to that which led to the kinematic connection we can prove that also the converse is true: the family of geodesics uniquely determines the symmetric connection. To find other solutions of the system (59), we therefore try to find metrics which have the same geodesics as metric (54). Alternatively, this process can be viewed

as the study of diffeomorphisms of $SE(3)$ which map geodesics to geodesics (such maps are called *affine maps*).

We saw earlier that a geodesic for the metric (54) between two points A_1 and A_2 on $SE(3)$ is a product of a geodesic on $SO(3)$ equipped with a bi-invariant metric (a constant velocity rotation) and a geodesic on Euclidean space \mathbb{R}^3 (a straight line). But on \mathbb{R}^3 , straight lines are geodesics for any inner product defined by a positive definite constant matrix. Therefore, any product metric on $SO(3) \times \mathbb{R}^3$ with the bi-invariant metric on $SO(3)$ and an inner product metric on \mathbb{R}^3 has the same geodesics as the metric (54). It can be shown [17] that for the basis \hat{L}_i such a metric has the form:

$$G(R, d) = \begin{bmatrix} \alpha I & 0 \\ 0 & R^T W R \end{bmatrix} \quad (55)$$

where W is a constant positive definite symmetric matrix which defines the inner product on \mathbb{R}^3 . We state this as a lemma:

Lemma 4.5 *If a metric G has the form (55), then it is a solution of the system of equations (59).*

Note that the form of G in Lemma 4.5 is consistent with that in Lemma 4.2. It is also obvious that the metric (54) can be obtained as a special case of (55) by substituting $W = \beta I$.

Lemma 4.5 identifies a family of Riemannian metrics that are compatible with the kinematic connection. We would like to know whether there are any other metrics which are compatible with the kinematic connection. Before we answer this question, we prove the following lemma:

Lemma 4.6 *Two metrics on $SE(3)$ which have the same Riemannian connection and are equal at a point, are equal everywhere.*

Proof: Let G_1 and G_2 be the two metrics and let A_0 be the point where they are equal: $G_1(A_0) = G_2(A_0)$. If a connection is compatible with the metric then the parallel transport preserves the inner product. Take an arbitrary point $A \in SE(3)$. We can always find a curve γ which connects A_0 with A . Since at A_0 , G_1 and G_2 are equal, we can choose a basis X_i for the tangent space $T_{A_0}SE(3)$ which is orthonormal in both metrics. The two metrics have the same connection and we can parallel transport vectors X_i at A_0 to vectors X'_i at A along γ . Since both metrics are compatible with the connection and the parallel transport preserves the inner product, vectors X'_i at A are orthonormal in both metrics. But this means that the two metrics are equal at A . Since A was an arbitrary point, the two metrics must be the same everywhere. \square

We can now state the second major result of the section:

Theorem 4.7 *A metric $G(R, d)$ is compatible with the kinematic connection given by Equation (45) and (46) if and only if it has the form:*

$$G(R, d) = \begin{bmatrix} \alpha I_{3 \times 3} & 0_{3 \times 3} \\ 0_{3 \times 3} & R^T W R \end{bmatrix} \quad (56)$$

where W is a constant positive definite symmetric matrix.

Proof: The “if” part of the Theorem is just Lemma 4.5. To prove the “only if” direction, let G be a metric compatible with the kinematic connection. According to Lemma 4.2, the metric G has the form:

$$G = \begin{bmatrix} \alpha I & 0 \\ 0 & M \end{bmatrix}, \quad (57)$$

where M is a positive definite, smoothly varying, symmetric matrix. At the identity there exists a metric G' of the form (56) with $W = M$ which is equal to G . The two metrics are both compatible with the kinematic connection, so according to Lemma 4.6, they are the same. Hence, the metric G has the form (56). \square

Remark 4.8 All the metrics of the form (56) are isometric. In other words, if G_1 and G_2 are two such metrics, there is an isometry between $SE(3)$ equipped with the metric G_1 and $SE(3)$ equipped with the metric G_2 . (This isometry does not preserve the group structure on $SE(3)$.) The two manifolds are isometric because of the product structure of the metrics and the fact that any two metrics on a Euclidean space \mathbb{R}^n obtained from an inner product are isometric.

Remark 4.9 Parallel transport along a closed curve γ which starts and ends at $A \in SE(3)$, maps an element $X \in T_A SE(3)$ to another element $X' \in T_A SE(3)$. The collection of such mappings that we obtain by taking all possible closed curves that start and end at A forms a group called *the holonomy group* of the connection with the reference point A . The derivation of Theorem 4.7 would be considerably shorter if we had used the de Rham splitting theorem and the structure of the holonomy group for the kinematic connection. However, we have elected to present a proof that only relies on elementary results from differential geometry.

5 Concluding remarks

There are many papers in kinematics and robotics that advocate the use of screw motions for the trajectory planning of robot end effectors (see for example, [12, 15, 16]). Another body of literature studies the trajectory planning problem on $SE(3)$ in the framework of Riemannian geometry and proposes the trajectories that are optimal with respect to some cost function [10, 18]. The focus of this paper is on how to choose a Riemannian metric and an affine connection on $SE(3)$ that are suitable for trajectory planning. To this end we systematically investigate the possible Riemannian metrics, affine connections, and the resulting geodesics (shortest distance trajectories) The key results and their implications for motion planning and kinematic analysis on $SE(3)$ are summarized below:

1. Screw motions are left or right translates of one-parameter subgroups (Proposition 3.2). Since one-parameter subgroups are a property of $SE(3)$ and are therefore independent of the choice of fixed or moving reference frames, it is attractive to use screw motions as the basis for trajectory planning. However, there is no positive definite cost function associated with the screw motion (Corollary 3.6).
2. If it is desirable to use screw motions in a particular application, it may be useful to think of the trajectories as extremals for a cost functional derived from the two-parameter family of semi-Riemannian metrics in Theorem 3.5. The semi-Riemannian metrics (and therefore the cost function) are bi-invariant.

3. When accelerations or higher order derivatives are necessary, either for trajectory planning [18] or for kinematic analysis [3], it is necessary to define a connection on $SE(3)$. Proposition 4.1 identifies the kinematic connection which through the covariant derivative yields the acceleration that is used in kinematics.
4. Theorem 4.7 identifies the family of Riemannian metrics that are compatible with the kinematic connection. This result can be used to formulate and solve the minimum acceleration or minimum jerk problems where functionals involving derivatives of the velocity are minimized to obtain optimal trajectories.
5. It is well-known that there is no bi-invariant metric on $SE(3)$ [5, 9]. The only family of metrics that would seem to make any sense for kinematic computations is thus the family defined in Theorem 4.7. If left-invariance is desired, Proposition 4.3 shows that the scale dependent left invariant metric of [10] is the only meaningful metric. This metric is independent of the choice of the fixed (inertial) reference frame⁶.

The results in this paper are of considerable theoretical significance and contribute to a deeper understanding of the trajectory generation problem on $SE(3)$. Further, they are directly relevant to applications in robotics, computer animation and computer aided design.

Acknowledgment

The financial support for this work has been provided by the NSF grants BCS 92-16691, MSS 91-57156, CISE/CDA 88-22719 and DMS 95-05175, ARPA Grant N00014-92-J-1647 and Army Grant DAAH04-96-1-0007. The first author was also supported in part by a fellowship (NSF grant SBR 89-20230) from the Institute for Research in Cognitive Science at the University of Pennsylvania.

⁶If right invariance is desired, or in other words, if we desire our metric to be independent of the choice of the moving (body-fixed) reference frame, an argument similar to the proof of Proposition 4.3 can be used to develop the appropriate right invariant metric.

A Equations defining metric with screw motions as geodesics

In Section 3.1 we concluded that Equation (24):

$$\widehat{L}_k(g_{ij}) = \frac{1}{2} \sum_l (C_{ki}^l g_{lj} + C_{kj}^l g_{li}),$$

must be satisfied by the metric if screw motions are geodesics. The coefficients C_{ij}^k are the structure constants of the Lie algebra $se(3)$. We evaluated this equation in Mathematica to obtain a system of 126 partial differential equations that have to be solved for the metric coefficients g_{ij} . In the equations we use the abbreviation $\mathcal{G}_{ij}^k \stackrel{\text{def}}{=} \widehat{L}_k(g_{ij})$.

$$\begin{aligned}
\mathcal{G}_{11}^1 &= 0 & \mathcal{G}_{11}^2 &= -g_{13} & \mathcal{G}_{11}^3 &= g_{12} \\
\mathcal{G}_{11}^4 &= 0 & \mathcal{G}_{11}^5 &= -g_{16} & \mathcal{G}_{11}^6 &= g_{15} \\
\mathcal{G}_{12}^1 &= \frac{1}{2}g_{13} & \mathcal{G}_{12}^2 &= -\frac{1}{2}g_{23} & \mathcal{G}_{12}^3 &= \frac{1}{2}(g_{22} - g_{11}) \\
\mathcal{G}_{12}^4 &= \frac{1}{2}g_{16} & \mathcal{G}_{12}^5 &= -\frac{1}{2}g_{26} & \mathcal{G}_{12}^6 &= \frac{1}{2}(g_{25} - g_{14}) \\
\mathcal{G}_{13}^1 &= -\frac{1}{2}g_{12} & \mathcal{G}_{13}^2 &= \frac{1}{2}(g_{11} - g_{33}) & \mathcal{G}_{13}^3 &= \frac{1}{2}g_{23} \\
\mathcal{G}_{13}^4 &= -\frac{1}{2}g_{15} & \mathcal{G}_{13}^5 &= \frac{1}{2}(g_{14} - g_{36}) & \mathcal{G}_{13}^6 &= \frac{1}{2}g_{35} \\
\mathcal{G}_{14}^1 &= 0 & \mathcal{G}_{14}^2 &= \frac{1}{2}(-g_{34} - g_{16}) & \mathcal{G}_{14}^3 &= \frac{1}{2}(g_{24} + g_{15}) \\
\mathcal{G}_{14}^4 &= 0 & \mathcal{G}_{14}^5 &= -\frac{1}{2}g_{46} & \mathcal{G}_{14}^6 &= \frac{1}{2}g_{45} \\
\mathcal{G}_{15}^1 &= \frac{1}{2}g_{16} & \mathcal{G}_{15}^2 &= -\frac{1}{2}g_{35} & \mathcal{G}_{15}^3 &= \frac{1}{2}(g_{25} - g_{14}) \\
\mathcal{G}_{15}^4 &= 0 & \mathcal{G}_{15}^5 &= -\frac{1}{2}g_{56} & \mathcal{G}_{15}^6 &= \frac{1}{2}g_{55} \\
\mathcal{G}_{16}^1 &= -\frac{1}{2}g_{15} & \mathcal{G}_{16}^2 &= \frac{1}{2}(g_{14} - g_{36}) & \mathcal{G}_{16}^3 &= \frac{1}{2}g_{26} \\
\mathcal{G}_{16}^4 &= 0 & \mathcal{G}_{16}^5 &= -\frac{1}{2}g_{66} & \mathcal{G}_{16}^6 &= \frac{1}{2}g_{56} \\
\mathcal{G}_{22}^1 &= g_{23} & \mathcal{G}_{22}^2 &= 0 & \mathcal{G}_{22}^3 &= -g_{12} \\
\mathcal{G}_{22}^4 &= g_{26} & \mathcal{G}_{22}^5 &= 0 & \mathcal{G}_{22}^6 &= -g_{24} \\
\mathcal{G}_{23}^1 &= \frac{1}{2}(g_{33} - g_{22}) & \mathcal{G}_{23}^2 &= \frac{1}{2}g_{12} & \mathcal{G}_{23}^3 &= -\frac{1}{2}g_{13} \\
\mathcal{G}_{23}^4 &= \frac{1}{2}(g_{36} - g_{25}) & \mathcal{G}_{23}^5 &= \frac{1}{2}g_{24} & \mathcal{G}_{23}^6 &= -\frac{1}{2}g_{34} \\
\mathcal{G}_{24}^1 &= \frac{1}{2}g_{34} & \mathcal{G}_{24}^2 &= -\frac{1}{2}g_{26} & \mathcal{G}_{24}^3 &= \frac{1}{2}(g_{25} - g_{14}) \\
\mathcal{G}_{24}^4 &= \frac{1}{2}g_{46} & \mathcal{G}_{24}^5 &= 0 & \mathcal{G}_{24}^6 &= -\frac{1}{2}g_{44} \\
\mathcal{G}_{25}^1 &= \frac{1}{2}(g_{35} + g_{26}) & \mathcal{G}_{25}^2 &= 0 & \mathcal{G}_{25}^3 &= \frac{1}{2}(-g_{15} - g_{24}) \\
\mathcal{G}_{25}^4 &= \frac{1}{2}g_{56} & \mathcal{G}_{25}^5 &= 0 & \mathcal{G}_{25}^6 &= -\frac{1}{2}g_{45} \\
\mathcal{G}_{26}^1 &= \frac{1}{2}(g_{36} - g_{25}) & \mathcal{G}_{26}^2 &= \frac{1}{2}g_{24} & \mathcal{G}_{26}^3 &= -\frac{1}{2}g_{16} \\
\mathcal{G}_{26}^4 &= \frac{1}{2}g_{66} & \mathcal{G}_{26}^5 &= 0 & \mathcal{G}_{26}^6 &= -\frac{1}{2}g_{46} \\
\mathcal{G}_{33}^1 &= -g_{23} & \mathcal{G}_{33}^2 &= g_{13} & \mathcal{G}_{33}^3 &= 0 \\
\mathcal{G}_{33}^4 &= -g_{35} & \mathcal{G}_{33}^5 &= g_{34} & \mathcal{G}_{33}^6 &= 0 \\
\mathcal{G}_{34}^1 &= -\frac{1}{2}g_{24} & \mathcal{G}_{34}^2 &= \frac{1}{2}(g_{14} - g_{36}) & \mathcal{G}_{34}^3 &= \frac{1}{2}g_{35} \\
\mathcal{G}_{34}^4 &= -\frac{1}{2}g_{45} & \mathcal{G}_{34}^5 &= \frac{1}{2}g_{44} & \mathcal{G}_{34}^6 &= 0 \\
\mathcal{G}_{35}^1 &= \frac{1}{2}(g_{36} - g_{25}) & \mathcal{G}_{35}^2 &= \frac{1}{2}g_{15} & \mathcal{G}_{35}^3 &= -\frac{1}{2}g_{34} \\
\mathcal{G}_{35}^4 &= -\frac{1}{2}g_{55} & \mathcal{G}_{35}^5 &= \frac{1}{2}g_{45} & \mathcal{G}_{35}^6 &= 0
\end{aligned} \tag{58}$$

$$\begin{array}{lll}
\mathcal{G}_{36}^1 = \frac{1}{2}(-g_{26} - g_{35}) & \mathcal{G}_{36}^2 = \frac{1}{2}(g_{16} + g_{34}) & \mathcal{G}_{36}^3 = 0 \\
\mathcal{G}_{36}^4 = -\frac{1}{2}g_{56} & \mathcal{G}_{36}^5 = \frac{1}{2}g_{46} & \mathcal{G}_{36}^6 = 0 \\
\mathcal{G}_{44}^1 = 0 & \mathcal{G}_{44}^2 = -g_{46} & \mathcal{G}_{44}^3 = g_{45} \\
\mathcal{G}_{44}^4 = 0 & \mathcal{G}_{44}^5 = 0 & \mathcal{G}_{44}^6 = 0 \\
\mathcal{G}_{45}^1 = \frac{1}{2}g_{46} & \mathcal{G}_{45}^2 = -\frac{1}{2}g_{56} & \mathcal{G}_{45}^3 = \frac{1}{2}(g_{55} - g_{44}) \\
\mathcal{G}_{45}^4 = 0 & \mathcal{G}_{45}^5 = 0 & \mathcal{G}_{45}^6 = 0 \\
\mathcal{G}_{46}^1 = -\frac{1}{2}g_{45} & \mathcal{G}_{46}^2 = \frac{1}{2}(g_{44} - g_{66}) & \mathcal{G}_{46}^3 = \frac{1}{2}g_{56} \\
\mathcal{G}_{46}^4 = 0 & \mathcal{G}_{46}^5 = 0 & \mathcal{G}_{46}^6 = 0 \\
\mathcal{G}_{55}^1 = g_{56} & \mathcal{G}_{55}^2 = 0 & \mathcal{G}_{55}^3 = -g_{45} \\
\mathcal{G}_{55}^4 = 0 & \mathcal{G}_{55}^5 = 0 & \mathcal{G}_{55}^6 = 0 \\
\mathcal{G}_{56}^1 = \frac{1}{2}(g_{66} - g_{55}) & \mathcal{G}_{56}^2 = \frac{1}{2}g_{45} & \mathcal{G}_{56}^3 = -\frac{1}{2}g_{46} \\
\mathcal{G}_{56}^4 = 0 & \mathcal{G}_{56}^5 = 0 & \mathcal{G}_{56}^6 = 0 \\
\mathcal{G}_{66}^1 = -g_{56} & \mathcal{G}_{66}^2 = g_{46} & \mathcal{G}_{66}^3 = 0 \\
\mathcal{G}_{66}^4 = 0 & \mathcal{G}_{66}^5 = 0 & \mathcal{G}_{66}^6 = 0
\end{array}$$

B Equations for the metric compatible with the kinematic connection

In Section 4 we concluded that a metric compatible with the kinematic connection must satisfy Equation (49):

$$\widehat{L}_k(g_{ij}) = \sum_l (\Gamma_{ik}^l g_{lj} + \Gamma_{jk}^l g_{li}).$$

The Christoffel symbols Γ_{ij}^k specify the kinematic connection and are listed in (47). For $k > 3$, the Christoffel symbols Γ_{ik}^j are 0, and therefore $\widehat{L}_k(g_{ij}) = 0$. For this reason, we only list equations for $k \leq 3$. The equations were generated in Mathematica and are listed below. In the equations, \mathcal{G}_{ij}^k stands for $\widehat{L}_k(g_{ij})$.

$$\begin{aligned}
\mathcal{G}_{11}^1 &= 0 & \mathcal{G}_{11}^2 &= -g_{13} & \mathcal{G}_{11}^3 &= g_{12} \\
\mathcal{G}_{12}^1 &= \frac{1}{2}g_{13} & \mathcal{G}_{12}^2 &= -\frac{1}{2}g_{23} & \mathcal{G}_{12}^3 &= \frac{1}{2}g_{22} - \frac{1}{2}g_{11} \\
\mathcal{G}_{13}^1 &= -\frac{1}{2}g_{12} & \mathcal{G}_{13}^2 &= \frac{1}{2}g_{11} - \frac{1}{2}g_{33} & \mathcal{G}_{13}^3 &= \frac{1}{2}g_{23} \\
\mathcal{G}_{14}^1 &= 0 & \mathcal{G}_{14}^2 &= -\frac{1}{2}g_{34} - g_{16} & \mathcal{G}_{14}^3 &= \frac{1}{2}g_{24} + g_{15} \\
\mathcal{G}_{15}^1 &= g_{16} & \mathcal{G}_{15}^2 &= -\frac{1}{2}g_{35} & \mathcal{G}_{15}^3 &= \frac{1}{2}g_{25} - g_{14} \\
\mathcal{G}_{16}^1 &= -g_{15} & \mathcal{G}_{16}^2 &= g_{14} - \frac{1}{2}g_{36} & \mathcal{G}_{16}^3 &= \frac{1}{2}g_{26} \\
\mathcal{G}_{22}^1 &= g_{23} & \mathcal{G}_{22}^2 &= 0 & \mathcal{G}_{22}^3 &= -g_{12} \\
\mathcal{G}_{23}^1 &= \frac{1}{2}g_{33} - \frac{1}{2}g_{22} & \mathcal{G}_{23}^2 &= \frac{1}{2}g_{12} & \mathcal{G}_{23}^3 &= -\frac{1}{2}g_{13} \\
\mathcal{G}_{24}^1 &= \frac{1}{2}g_{34} & \mathcal{G}_{24}^2 &= -g_{26} & \mathcal{G}_{24}^3 &= g_{25} - \frac{1}{2}g_{14} \\
\mathcal{G}_{25}^1 &= \frac{1}{2}g_{35} + g_{26} & \mathcal{G}_{25}^2 &= 0 & \mathcal{G}_{25}^3 &= -\frac{1}{2}g_{15} - g_{24} \\
\mathcal{G}_{26}^1 &= \frac{1}{2}g_{36} - g_{25} & \mathcal{G}_{26}^2 &= g_{24} & \mathcal{G}_{26}^3 &= -\frac{1}{2}g_{16} \\
\mathcal{G}_{33}^1 &= -g_{23} & \mathcal{G}_{33}^2 &= g_{13} & \mathcal{G}_{33}^3 &= 0 \\
\mathcal{G}_{34}^1 &= -\frac{1}{2}g_{24} & \mathcal{G}_{34}^2 &= \frac{1}{2}g_{14} - g_{36} & \mathcal{G}_{34}^3 &= g_{35} \\
\mathcal{G}_{35}^1 &= g_{36} - \frac{1}{2}g_{25} & \mathcal{G}_{35}^2 &= \frac{1}{2}g_{15} & \mathcal{G}_{35}^3 &= -g_{34} \\
\mathcal{G}_{36}^1 &= -\frac{1}{2}g_{26} - g_{35} & \mathcal{G}_{36}^2 &= \frac{1}{2}g_{16} + g_{34} & \mathcal{G}_{36}^3 &= 0 \\
\mathcal{G}_{44}^1 &= 0 & \mathcal{G}_{44}^2 &= -2g_{46} & \mathcal{G}_{44}^3 &= 2g_{45} \\
\mathcal{G}_{45}^1 &= g_{46} & \mathcal{G}_{45}^2 &= -g_{56} & \mathcal{G}_{45}^3 &= g_{55} - g_{44} \\
\mathcal{G}_{46}^1 &= -g_{45} & \mathcal{G}_{46}^2 &= g_{44} - g_{66} & \mathcal{G}_{46}^3 &= g_{56} \\
\mathcal{G}_{55}^1 &= 2g_{56} & \mathcal{G}_{55}^2 &= 0 & \mathcal{G}_{55}^3 &= -2g_{45} \\
\mathcal{G}_{56}^1 &= g_{66} - g_{55} & \mathcal{G}_{56}^2 &= g_{45} & \mathcal{G}_{56}^3 &= -g_{46} \\
\mathcal{G}_{66}^1 &= -2g_{56} & \mathcal{G}_{66}^2 &= 2g_{46} & \mathcal{G}_{66}^3 &= 0
\end{aligned} \tag{59}$$

References

- [1] R. L. Bishop and S. I. Goldberg. *Tensor Analysis and Manifolds*. Dover Publications, New York, 1980.
- [2] M. P. do Carmo. *Riemannian geometry*. Birkhauser, Boston, 1992.
- [3] R. Featherstone. *Robot Dynamics Algorithms*. Kluwer Academic Publishers, 1987.

- [4] J. M. Herve. Intrinsic formulation of problems of geometry and kinematics of mechanisms. *Mechanism and Machine Theory*, 17(3):179–184, 1982.
- [5] A. Karger and J. Novak. *Space Kinematics and Lie Groups*. Gordon and Breach Science Publishers, 1985.
- [6] K.H.Hunt. *Kinematic Geometry of Mechanisms*. Clarendon Press, Oxford, 1978.
- [7] S. Kobayashi and Nomizu K. *Foundations of Differential Geometry*, volume 1. Interscience, New York-London, 1962.
- [8] J. M. McCarthy. *An Introduction to Theoretical Kinematics*. MIT Press, 1990.
- [9] R. M. Murray, Z. Li, and S. S. Sastry. *A Mathematical Introduction to Robotic Manipulation*. CRC Press, 1994.
- [10] F. C. Park. Distance metrics on the rigid-body motions with applications to mechanism design. *ASME Journal of Mechanical Design*, 117(1):48–54, 1995.
- [11] F. C. Park and R. W. Brockett. Kinematic dexterity of robotic mechanisms. *International Journal of Robotics Research*, 13(1):1–15, 1994.
- [12] D. L. Pieper. *The kinematics of manipulators under computer control*. PhD thesis, Stanford University, 1968.
- [13] D.H. Sattinger and O.L. Weaver. *Lie groups and algebras with applications to physics*. Springer-Verlag, New York, 1986.
- [14] B. F. Schutz. *Geometrical Methods of Mathematical Physics*. Cambridge University Press, Cambridge, 1980.
- [15] K. J. Waldron. Geometrically based manipulator rate control algorithms. *Mechanism and Machine Theory*, 17(6):379–385, 1982.
- [16] D. E. Whitney. The mathematics of coordinated control of prosthetic arms and manipulators. *ASME Journal of Dynamic Systems, Measurement, and Control*, 94:303–309, 1972.
- [17] M. Žefran. *Continuous methods for motion planning*. PhD thesis, U. of Pennsylvania, Philadelphia, PA, 1996.
- [18] M. Žefran, V. Kumar, and C. Croke. On the generation of smooth three-dimensional rigid body motions. Submitted to IEEE Transactions on Robotics and Automation, 1995.