An investigation into non-smooth locomotion

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Abstract

We analyze a class of mechanisms that locomote by switching between constraints. Because of the hybrid nature of such systems, most of the existing analysis tools, developed primarily for smooth systems, can not be directly applied. Our aim is to exploit the special structure provided by Lagrangian mechanics to study the controllability of this class of mechanisms. We base the analysis on a series representation of the evolution of the system. Our main result is a description of trajectories involving switches between constraints at nonzero velocity (impacts) in the presence of large inertial forces (drift). The analysis provides a basis for local motion planning. The results are applied to an example of a two-link planar mechanism that can locomote by clamping one of the links.

Keywords: mechanical control systems, hybrid systems, controllability, series expansions

1 Introduction

Locomotion devices belong to the class of mechanical control systems (Lagrangian systems) and have a special structure that can be fruitfully exploited for control. Advances in the control of Lagrangian systems have led to several important results when the mechanics of locomotion is smooth [1, 2, 3]. On the other hand, several locomotion modalities, including legged locomotion involve impacts and switches in the dynamic behavior and are therefore inherently non-smooth.

The focus of this paper is on the reachability analysis of non-smooth locomotion devices. Such analysis is instrumental for design of motion planning and control algorithms. We limit our study to a class of systems that can locomote by switching between constraints. An example is a planar linkage sliding on a frictionless surface that can arbitrarily clamp a subset of its links to the surface. While the systems considered in this paper are simpler than legged devices, we hope that such structured examples can provide insight into more general problems.

The underlying mathematical framework of this paper is Riemannian geometry as it applies to mechanical control systems [4]. Modeling and controllability results for smooth mechanical systems are discussed in [5]. Of special importance in the controllability analysis are series describing the evolution of the system [6, 7]. Such series are also a basis of several motion planning algorithms [8, 9, 10].

Non-smooth locomotion systems belong to the class of hybrid systems. While a number of approaches to modeling and control of hybrid systems exist [11, 12], these works are in general not applicable to locomotion. Some exceptions are [13] which provides a quasi-static controllability analysis of locomotion devices, and [14] where a method for stabilizing systems with changing dynamics is described.

The paper is organized as follows. We start with a review of Lagrangian dynamics in the Riemannian setting. Then we formally define hybrid mechanical control systems. In the following section we describe a series for the evolution of a control system. We specialize the series to mechanical systems and show how a sufficient condition for controllability of hybrid mechanical control systems can be obtained. Finally, we apply the results to a simple two-link device and show that it can locomote by clamping one of the links.

2 Mechanical control systems

Let Q be the configuration manifold of the system with coordinates $q = (q_1, \dots, q_n)$. At every $q \in Q$, the kinetic energy of the system defines a Riemannian metric, M_q. Hamilton's principle states that unforced motions of the system correspond to geodesics with respect to the metric M[1] and are thus given by the geodesic equation:

$$\nabla_{\dot{q}} \dot{q} = 0 \tag{1}$$

where ∇ is the *Levi-Civita connection* corresponding to M [15]. In coordinates (we use the summation convention throughout the paper), an affine connection is given by:

$$\nabla_X Y = \left(\frac{\partial Y^i}{\partial q^j} X^j + \Gamma^i_{jk} X^j Y^k \right) \frac{\partial}{\partial q^i} \tag{2}$$

where the coefficients Γ^i_{jk} are known as Christoffel symbols.

[1] The controllability analysis is simpler if the potential forces are present so we will neglect them in this work.

When an external force (a one-form) F acts on the system, the dynamic equations take the form:

$$\nabla_{\dot{q}}\dot{q} = Y \tag{3}$$

where $Y(q) = M_q^{-1}F(q)$ is now a vector field. Written in coordinates, these *forced Euler-Lagrange* equations take the familiar form:

$$\ddot{q} + M_q^{-1}C(q,\dot{q}) = M_q^{-1}F \tag{4}$$

where M_q is the inertia matrix and $C(q,\dot{q})$ are the Coriolis and centrifugal forces. Formally a *mechanical control system* can be defined as a tuple $(Q, M, \mathcal{F}, U)$, where Q is an n-dimensional configuration manifold, M is a Riemannian metric on Q (the kinetic energy), $\mathcal{F} = \text{span}\{F^1, \dots, F^m\}$ is the input co-distribution (input forces), and $U \subset \mathbb{R}^m$ is the set of inputs. Later in the paper we will also need the notion of input distribution, $\mathcal{Y} = \text{span}\{M^{-1}F^i\}$.

2.1 Constraints

From the control point of view, constraints on a mechanical system limit the set of directions in which the system can move. Therefore, an intrinsic description of a constraint is a distribution on Q, describing at each point the set of feasible velocities. Such description applies both to holonomic and nonholonomic constraints.

A mechanical control system together with a constrained distribution will be called a *constrained mechanical control system*, $\Sigma = (Q, M, \mathcal{F}, \mathcal{D}, U)$. It was shown in [16] that the dynamic equations for such a system can be also written using an affine connection:

$$\widetilde{\nabla}_{\dot{q}}\dot{q} = u^k P_{\mathcal{D}}(Y^k), \tag{5}$$

where $P_{\mathcal{D}} : TQ \to TQ$ is the orthogonal projection to $\mathcal{D}$ (with respect to the metric M) and $\widetilde{\nabla}$ is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \left(\nabla_X(I - P)\right)(Y), \qquad \forall X, Y.$$

Since equation (5) is formally identical to equation (3), such description provides a unified treatment of both constrained and unconstrained mechanical control systems, see [16].

2.2 Hybrid mechanical control systems

A *hybrid mechanical control system* consists of a mechanical control system $(Q, M, \mathcal{F}, U)$ together with a given set of constraint distributions $\mathcal{D}_i$, where i belongs to an index set I. Each constraint $\mathcal{D}_i$ yields a constrained mechanical control system $\Sigma_i = (Q, M, \mathcal{F}, U, \mathcal{D}_i)$, with associated affine connection ∇_i and input distribution $\mathcal{Y}_i = \text{span}\{P_{\mathcal{D}_i}M^{-1}F\}$. Formally, the hybrid mechanical control system is therefore a tuple $(I, Q, M, \mathcal{F}, U, \{\mathcal{D}_i\}_{i\in I})$. Slightly more general definition can be found in [17].

The evolution of a hybrid mechanical control system can be described as follows. The system starts in a state $((q,\dot{q}), i) \in TQ \times I$ and it evolves according to the dynamics given by ∇_i and the chosen set of controls. At any point, we can choose to switch to any other discrete state. Whenever the system switches between two discrete states i and j (constraint distributions $\mathcal{D}_i$ and $\mathcal{D}_j$) it undergoes impact. The velocity after the impact is the orthogonal projection onto $\mathcal{D}_j$ of the velocity before the impact:

$$\dot{q}(t^+) = P_{\mathcal{D}_j}\dot{q}(t^-) \tag{6}$$

3 Series for evolution of a control system

In this section we give an overview of two series describing the evolution of a control system. The main references are [6] and [18]. Note that these results are not limited to mechanical control systems.

Consider a control system evolving on a manifold M:

$$\dot{x}(t) = f_0(x) + \sum_{i=1}^{m} u_i f_i(x) \tag{7}$$

where f_i are vector fields on M and $u = (u_1, \dots, u^m)$ is an integrable function $u : [0,T] \to \mathbb{R}^m$. Let $x(t, u; x_0)$ be a solution of (7) for the initial condition $x(0) = x_0$. Our purpose is to characterize $x(t, u; x_0)$. For what follows, it is important to remember that a vector field represents a differentiable operator on the set of smooth functions $C^\infty(M)$ of M to $\mathbb{R}$. For a multi-index $I = (i_1, \dots, i_r)$, where $0 \le i_j \le m$, we can therefore define an operator $f_I = f_{i_1} \circ f_{i_2} \circ \dots \circ f_{i_r}$ (for simplicity we will omit the symbol $\circ$ in the rest of the paper). For a multi-index I we can also define:

$$\int_0^t u_I = \int_0^t u_{i_r}(t_r) \int_0^{t_r} \cdots \int_0^{t_2} u_{i_1}(t_1)dt_1 dt_2 \cdots dt_r$$

Now consider a formal series:

$$Ser_f(T,u) = \sum_I \left(\int_0^T u_I\right) f_I, \tag{8}$$

where the summation is over all possible multi-indices I (including the zero-length index e with $f_e = \text{Id}$). This formal series is known as *Chen-Fliess series*. It was shown in [7] that if the vector fields f_i are analytic and $Ser_f(T,u)$ is applied to an analytic function $\Phi \in C^\infty(M)$, then there exists $T > 0$ such that for every $t < T$:

$$Ser_f(t,u)\Phi|_{x_0} = \Phi(x(t,u;x_0)) \tag{9}$$

The problem with the Chen-Fliess series is that it is difficult to identify operators of a certain order that get applied to Φ. In analogy with the Taylor series, we would like to be able to say what is the term in the series of order 0, 1, and so on. The problem becomes apparent if we consider second-order operators $f_{i_1}f_{i_2}$ and $f_{i_2}f_{i_1}$: their linear combination $f_{i_1}f_{i_2} - f_{i_2}f_{i_1} = [f_{i_1}, f_{i_2}]$ is a first-order operator.

In [6] a series is described that avoids this problem. Define the operator exponential:

$$\exp(f) = \sum_{i=0}^{\infty} \frac{f^i}{i!} \tag{10}$$

and let $[f_I] \overset{\text{def}}{=} [f_{i_1}, [f_{i_2}, \cdots, [f_{i_{r-1}}, f_{i_r}] \cdots]]$. Let $|I|$ denote the length of the multi-index I. Then it can be shown that:

$$Ser_f(T,u) = \exp\left(Z_f(T,u)\right) \tag{11}$$

where

$$Z_f(T,u) = \sum_{n=1}^{\infty} \sum_{I_i \neq e} C_{I_1 \cdots I_n} [f_{I_1 \cdots I_n}] \tag{12}$$

and the coefficient $C_{I_1 \cdots I_n}$ is given by:

$$C_{I_1 \cdots I_n} = \frac{(-1)^n}{n\left(|I_1| + \cdots + |I_n|\right)} \left(\int_0^T u_{I_1}\right) \cdots \left(\int_0^T u_{I_n}\right).$$

Since there is exactly one operator of every order in the exponential series (10) and since all the terms in the series (12) are first-order, the series (11) clearly has the property we were looking for. This series plays an important role in the controllability analysis, as was shown in [7].

3.1 Evolution of a mechanical control system

In this section we examine the brackets that appear in $Z_f(T,u)$ for a mechanical control system. To this end, we will exploit the structure of the equations (5). To use the series (11) we have to rewrite the second-order equations (5) in the configuration variables q as a system of first-order equations in the variables $q \in Q$ and $v \in TqQ$. At a point $(q,v) \in TTQ$, we define two operations $V_v, H_v : T_qQ \to T_{(q,v)}TQ$ as follows:

$$\begin{aligned} \mathrm{V}_v\left(X^i \frac{\partial}{\partial q^i}\right) &= X^i \frac{\partial}{\partial v^i} \\ \mathrm{H}_v\left(X^i \frac{\partial}{\partial q^i}\right) &= X^i \left(\frac{\partial}{\partial q^i} - \frac{1}{2}(\Gamma^k_{ij} + \Gamma^k_{ji}) v^j \frac{\partial}{\partial v^k}\right) \end{aligned} \tag{13}$$

Although they are written in coordinates, it can be shown that the definitions are intrinsic in the sense that they only depend on the affine connection, not the choice of coordinates.

To study the evolution of the system in the configuration variables, we will also make use of the natural projection $\tau_Q : TQ \to Q$, which is in coordinates $\tau_Q((q,v)) = q$. By taking the tangent map $T\tau_Q$, we can then project vectors from $T_{(q,v)}TQ$ to T_qQ. It is not difficult to see that the operators $\mathrm{H}_v(v)$ and $\mathrm{V}_v(Y_i)$ satisfy:

$$\begin{aligned} T\tau_Q\, \mathrm{H}_v(Y) &= Y, \\ T\tau_Q\, \mathrm{V}_v(Y) &= 0. \end{aligned}$$

Using these definitions we rewrite the equations of motion

$$\nabla_v v = u^i Y_i$$

into the form:

$$\frac{d}{dt}\begin{bmatrix} q \\ v \end{bmatrix} = \mathrm{H}_v(v) + u^i\, \mathrm{V}_v(Y_i)\,. \tag{14}$$

For the latter control system we compute the Lie brackets of order up to three. By direct computation we can verify that in the "configuration component" the only non-vanishing terms for the second and third order brackets are:

$$\begin{aligned} T\tau_Q\left[\mathrm{H}_v(v)\,, \mathrm{V}_v(Y)\right] &= -Y, \\ T\tau_Q[\mathrm{V}_v(Y_i)\,, [\mathrm{H}_v(v)\,, \mathrm{V}_v(Y_j)]] &= 0, \\ T\tau_Q[\mathrm{H}_v(v)\,, [\mathrm{H}_v(v)\,, \mathrm{V}_v(Y_i)]] &= -2\nabla_v Y_i. \end{aligned} \tag{15}$$

4 Equilibrium controllability

In the case of mechanical control systems two controllability questions can be asked: (a) what points in the *phase space* (i.e., configuration and velocity) can be reached; and (b) what *configurations* can be reached. The first question can be addressed using standard nonlinear controllability methods. We will show that in the case of hybrid mechanical control systems, the second question is particularly interesting. For smooth systems, the question was posed and answered in [5] and we briefly review this work here.

Consider a system described by (5). Let q_0 be a point in Q and let W be a neighborhood of q_0. The reachable set of q_0 within W is

$$\mathcal{R}^W_Q(q_0, \leq T) = \cup_{\tau \leq T}\{x \in Q \mid \exists \text{ a solution } q(t) \text{ to } (5) \text{ s.t. } \dot{q}(0) = 0,\ q(t) \in W \text{ for } t \in [0,\tau],\ q(\tau) = x\}.$$

Note that the definition of $\mathcal{R}^W_Q(q_0, \leq T)$ only involves the configurations, not the velocities. This set is therefore different from a reachable set for the system (14) (which is a subset of TQ, not Q). The system (5) is *locally configuration controllable* at q_0 if there exists a time T such that $\mathcal{R}^W_Q(q_0, \leq T)$ contains a neighborhood of q_0 for any neighborhood W of q_0, and *equilibrium controllable* on $W \subset Q$, if for any two equilibrium points $q_1, q_2 \in W$, there exists an input $\{u^k(t), t \in [0,T]\}$ and a solution $\{q(t), t \in [0,T]\}$ such that $q(0) = q_1$, $q(T) = q_2$, $q(t) \in W$ for all $t \in [0,T]$, and $\dot{q}(0) = 0$, $\dot{q}(T) = 0$.

To characterize the configuration controllability, we introduce the following operations. As in [5], we define the *symmetric product* of two vector fields as:

$$\langle X : Y \rangle = \nabla_X Y + \nabla_Y X.$$

Let $\mathcal{X} = \{X_1, \ldots, X_m\}$ be a family of vector fields. We define $\mathrm{Lie}(\mathcal{X})$ to be the closure of $\mathcal{X}$ under the Lie bracket operation (the involutive closure), and we let $\mathrm{Sym}(\mathcal{X})$ be the closure of $\mathcal{X}$ under the symmetric product operation. Within the set $\mathrm{Sym}(\mathcal{X})$, we define the *order* of a symmetric product to be the number of vector fields X_j present in it. We say that a symmetric product is *bad* if it contains an even number of each X_i. Otherwise the product is said to be *good*. The controllability tests are then:

Theorem 4.1 ([5]) *The system is configuration controllable at q_0 if:*

(i) the rank of $\mathrm{Lie}(\mathrm{Sym}(\mathcal{Y}))$ is full;

(ii) at q_0, every bad symmetric product is a linear combination of lower order good symmetric products.

If these conditions are verified at every $q \in W$, then the system is equilibrium controllable on W.

4.1 Hybrid mechanical control systems

In [17] we studied configuration and equilibrium controllability for hybrid mechanical control systems. Since the analysis above requires computation of the brackets at zero velocity, we had to restrict our analysis to the case when the switches between different regimes occur at zero velocity. We showed:

Proposition 4.2 ([17]) *A hybrid mechanical control system is equilibrium controllable on an open set W if the following two conditions hold:*

(i) in each discrete state i, every bad symmetric product is a linear combination of lower order good symmetric products

(ii) the rank of $\mathrm{Lie}(\sum_{i\in I}\mathrm{Sym}_i(\mathcal{Y}_i))(q)$ is full for all $q \in W$.

Restricting a hybrid mechanical control system to switches at zero velocity seems overly restrictive. In the rest of the section we therefore generalize the above result to switches at nonzero velocity. The key observation is that if in some regime i the system can generate a nonzero velocity at q_0 (by making a small loop in the configuration space), then we can use this nonzero velocity to exploit directions generated by the brackets that would otherwise vanish. These directions are given precisely by the terms in equation (15).

The details of the computations are not as important as the fact that these additional terms can give us directions that can not be generated by starting at zero velocity and that curvature plays a role in computing these directions. Note that the above terms must be evaluated only for those values of v that can be generated by making small loops from q_0. These directions are characterized by $\mathrm{Sym}(\mathcal{Y})$, see [5].

By using brackets of higher order, we can generate more and more terms, but to simplify the presentation, we will not consider these here. For simplicity we will also assume that the system only has two regimes. Assume therefore a mechanical control system $(Q, M, \mathcal{F}, U)$ and a constrained distribution $\mathcal{D}$, forming a hybrid mechanical control system $\Sigma = (\{1,2\}, Q, M, \mathcal{F}, U, \{TQ, \mathcal{D}\})$ (the system can switch between free motion and motion in directions $\mathcal{D}$).

Proposition 4.3 *The system Σ is equilibrium controllable on some neighborhood W of q_0 if the following conditions hold:*

(i) For $k = 1, 2$, $\mathrm{Sym}(\{Y_i^1\}) \cup \mathrm{Sym}(\{Y_i^2\}) \cup \mathcal{D}^\perp = TQ$, where $Y_i^1 = Y_i$ and $Y_i^2 = P_{\mathcal{D}} Y_i^1$.

(ii) For $k = 1, 2$, the bad symmetric products $\langle Y_i^k : Y_i^k \rangle$ are spanned by the vector fields Y_i^k.

(iii) For every $q \in W$:

$$TQ = \mathrm{span}\Big\{\mathcal{D} \cup \{Y_i^k, [Y_i^k, Y_j^l]\} \cup \{\nabla_v Y_i^k, \nabla^2 Y_i^k(v,v) + R(v, Y_i^k)v\}_{v\in \mathrm{Sym}(\{Y_i^l\})}\Big\}$$

where k and l must be different regimes.

Proof: Condition (ii) corresponds to condition (i) in Proposition 4.2, while condition (iii) is similar to condition (ii) there, with added terms that can be generated by switches at nonzero velocity. These terms are evaluated on the set of velocities that can be generated in regime 2. Also, in the brackets from which these terms arise, each vector field Y_i^1 only appears once, so the sign of the term C_I in the series (12) can be changed by using $-u$ instead of u.

Conditions (ii) and (iii) imply only local configuration controllability, which means that we would not necessarily be able to stop after we moved to the desired configuration. However, if condition (i) holds, we can bring the velocity to zero at the final position by performing loops there. First, we bring to 0 those directions that are in $\mathrm{Sym}(\{Y_i^1\})$, then we switch to regime 2 and thus annihilate the directions in $\mathcal{D}^\perp$ through impact, and we finally perform a loop in regime 2 that brings to zero the velocity in directions $\mathrm{Sym}(\{Y_i^2\})$. ■

5 Example: Sliding and clamping mechanism

In this section we apply our analysis to a two-link mechanism sliding without friction on a plane. The mechanism consists of two homogeneous bars of unit density and lengths (l_1, l_2), connected by an actuated rotational joint (Figure 1). In the figure, CM denotes the center of mass of the two body system. The coordinates of the center of mass of the link j are (x_j, y_j, θ_j), while $(x_{\mathrm{CM}}, y_{\mathrm{CM}})$ are the coordinates of CM. One of the

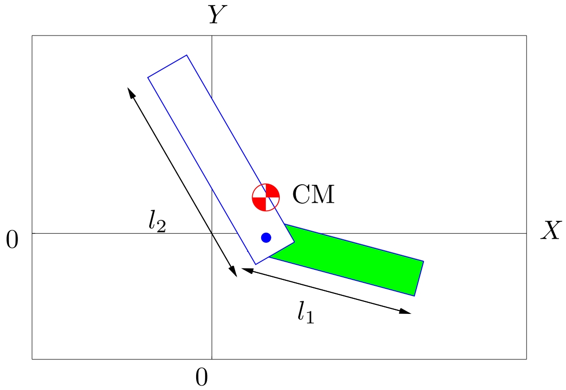

Figure 1: Sliding and clamping mechanism with two links. One of the links can be clamped to the floor so that it is completely immobilized.

links can be instantaneously clamped to the ground (anywhere on the plane) so that it gets completely immobilized. When the link is clamped, the number of degrees of freedom of the system decreases from 4 to 1. The clamping constraint is described by the function $\varphi_1(q) = (x_1, y_1, \theta_1)$.

The configuration manifold of the two body system is $Q = SE(2) \times S^1$. We will describe the configuration with the coordinates $q = (x_{\mathrm{CM}}, y_{\mathrm{CM}}, \theta, \phi)$, where $\theta = \theta_1$ and $\phi = \theta_2 - \theta_1$.

When the first link is clamped, the system is confined to the submanifold $R_1(q_0) = \{q \in Q \mid \varphi_1(q) = (x_0, y_0, \theta_0)\}$. This holonomic constraint induces the constraint distribution:

$$\mathcal{D}_2(q) = \operatorname{span}\left\{-2l_2^2 \sin(\phi+\theta)\frac{\partial}{\partial x_{\mathrm{CM}}} + 2l_2^2\cos(\phi+\theta)\frac{\partial}{\partial y_{\mathrm{CM}}} + 4(l_1+l_2)\frac{\partial}{\partial \phi}\right\}$$

If we set $\mathcal{D}_1(q) = T_q Q$, we thus have a hybrid mechanical control system $\Sigma = (\{1,2\}, Q, M, \mathcal{F}, U, \{\mathcal{D}_1, \mathcal{D}_2\})$. The input vector field is:

$$Y_0 = l_2^2\left(l_2(5l_1 + 2l_2) + 3l_1^2\cos(\phi)\right)\frac{\partial}{\partial\theta} - \left(2l_1^4 + 5l_1^3 l_2 + 5l_1 l_2^3 + 2l_2^4 + 6l_1^2 l_2^2\cos(\phi)\right)\frac{\partial}{\partial\phi}$$

The input vector field on the constrained regime is obtained by projecting Y_0 onto the constraint distribution. Since the constrained distribution is one-dimensional, the constrained input spans exactly the same direction as the constrained distribution.

We now have all the necessary tools to check for equilibrium controllability. Notice that in the regime 2 the system is fully controllable, $\mathrm{Sym}(\{Y_i^2\}) = \mathcal{D}$, so the condition (i) is trivially satisfied. We can also show that $\langle Y_0 : Y_0\rangle = \zeta Y_0$ for some scalar function ζ, so that condition (ii) is satisfied. Finally, we can check that

$$\operatorname{rank}\{Y_0, Y_1, [Y_0, Y_1], T\tau_Q[\mathrm{H}_v(v), [\mathrm{H}_v(v), [\mathrm{H}_v(v), \mathrm{V}_v(Y_0)]]]\}(q) = 4$$

in a neighborhood of the point $(x_{\mathrm{CM}}, y_{\mathrm{CM}}, \theta, \phi) = (0,0,0,0)$. Therefore, the hybrid mechanical control system Σ is equilibrium controllable.

It turns out that we could replace the last vector with the bracket $[Y_0, [Y_1, Y_0]]$ and still span the whole space. This means that the system is equilibrium controllable even if we only allow switches at zero velocity, as was shown in [17]. However, this bracket is really $[Y_0, [Y_1, Y_0]] = T\tau_Q[[\mathrm{H}_v(v), \mathrm{V}_v(Y_0)], [[\mathrm{H}_v(v), \mathrm{V}_v(Y_1)], [\mathrm{H}_v(v), \mathrm{V}_v(Y_0)]]]$, which is of higher order than the bracket generated by the switches at nonzero velocity. Therefore, even if switching at nonzero velocity does not contribute any new controllable directions, it allows us to move in certain directions much more efficiently.

Figure 2 shows a motion in the $\frac{\partial}{\partial x_{\mathrm{CM}}}$ and $\frac{\partial}{\partial y_{\mathrm{CM}}}$ directions generated by switching at nonzero velocity. The mechanism is initially clamped. It then swings the unclamped link to build the velocity, after which it releases the constraint and starts drifting. At the end, a maneuver is performed that stops the mechanism.

6 Conclusions

We investigated a class of mechanisms that can locomote by switching between constraints. The nonsmooth nature of such systems due to changes in the dynamic equations and impacts prevents the application of conventional tools for motion planning and control. However, we showed that the evolution of such systems can be studied using an operator series. By exploiting the special Lagrangian structure, we were able to characterize the terms in this series and relate them to the controllability of the system. In particular, we characterized the terms when the system switches between constraints at nonzero velocity. We showed that switches at nonzero velocity can provide new controllable directions. Furthermore, they can generate motions in certain directions more efficiently than gaits provided by the zero velocity analysis. The two crucial aspects of our analysis are that the systems we study contain drift and that we study their evolution at nonzero velocity.

This work opens several possible directions for future work. Most importantly, we plan to develop motion planning methods based on methods developed for smooth systems in [10, 9, 8]. Further, our characterization of the evolution of a mechanical control system at nonzero velocity could be useful in the controllability analysis for relative equilibria of underwater and aerospace vehicles. Finally, the present analysis motivates the investigation of optimal trajectories for the class of systems described in this paper.

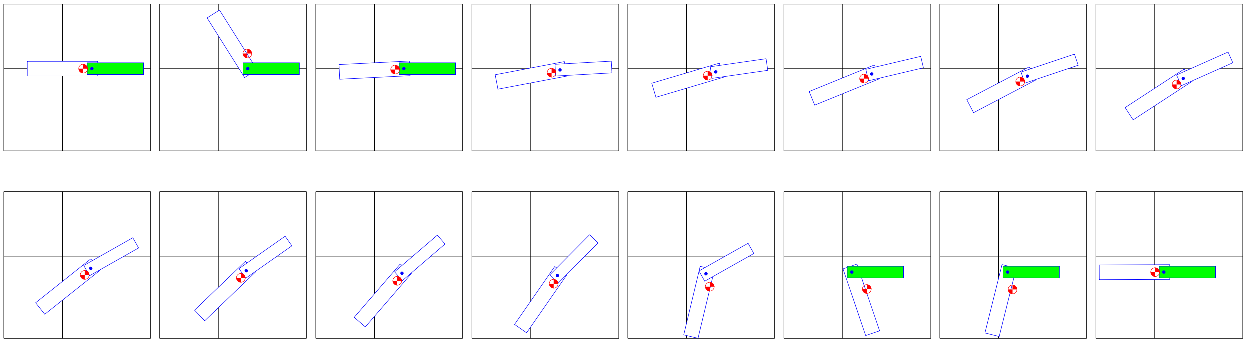

Figure 2: Motion generated by switch at nonzero velocity. The mechanism performs a maneuver at the beginning to build the velocity, then drifts, and at the end performs a maneuver to stop.

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