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Two methods for interpolating rigid body motions

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Abstract

This paper investigates methods for computing a smooth motion that interpolates a given set of positions and orientations of a rigid body. To make the interpolation independent of the representation of the motion, we use the coordinate-free framework of differential geometry. Inertial and body-fixed reference frames must be chosen to describe the position and orientation of the rigid body. We show that trajectories that are independent of the choice of these frames can be obtained by using the exponential map. Since these trajectories may exhibit rapid changes in velocity or its higher derivatives, a method for finding the maximally smooth interpolating curve is developed. Trajectories computed by both methods are compared on an example.

1 Introduction

The problem of finding a smooth motion that interpolates a given set of positions and orientations is frequently encountered in robotics (planning of trajectories in welding and painting), computer graphics (animation, interpolation of motion through a set of key frames) and computer-aided geometric design (interactive interpolation schemes for design), and has recently received considerable attention. The problem is well understood in Euclidean spaces [6, 9], but it is not clear how these techniques can be generalized to curved spaces. There are several issues that need to be addressed, particularly on non-Euclidean spaces. It is desirable that the computational scheme be independent of the description of the space and invariant with respect to the choice of the coordinate systems used to describe the motion. A method for computing points on the curve and the algebraic properties of the trajectories are also important. Finally, the interpolation properties of the trajectories and their smoothness properties need to be considered.

Shoemake [19] proposed a spherical analog of the de Casteljau algorithm to interpolate rotations with Bezier curves. This idea was extended by Ge and Ravani [8] and Park and Ravani [17] to spatial motions. Ge and Ravani [7] used the dual unit quaternion representation of $SE(3)$ and subsequently applied Euclidean methods to interpolate in this space. Jütler [10] formulated a more general version of the polynomial interpolation by using dual (instead of dual unit) quaternions to represent $SE(3)$. Kim *et al.* [12] generalized different polynomial interpolation schemes from Euclidean space to $SO(3)$. Park and Kang [16] derived a rational interpolating scheme for the group of rotations $SO(3)$. Several researchers generalized splines to the curved spaces by observing that in the Euclidean space they have certain minimizing properties. Noakes *et al.* [14] derived the necessary conditions for cubic splines on general manifolds. These results are extended in [4] to the dynamic interpolation problem. Necessary conditions for higher-order splines are derived in Camarinha *et al.* [3]. In [21, 22], the variational approach was applied to interpolation of smooth motions on $SE(3)$ and several metrics suitable for trajectory planning were identified.

In this paper, we use the results from [21, 22] to investigate various motion interpolation schemes on $SE(3)$. We pursue a geometric approach and require that our results be invariant with respect to the choice of reference frames and independent of the parameterization of the manifold. Our goal is to develop efficient methods for finding trajectories that minimize meaningful performance functions. However, we will show that there is a trade-off between the computational efficiency and optimality of the generated trajectories. We first describe a simple scheme for finding an interpolating motion that does not depend on the choice of the inertial and body-fixed reference frames. The scheme is based on the properties of the exponential map on $SE(3)$. Next, we outline a method for computing minimum acceleration curves on $SE(3)$, which are a generalization of cubic splines. This scheme follows the approach in [21]. We compute the curves for two choices of the metric on $SE(3)$ and compare them with the trajectories obtained by the scheme based on the exponential map.

2 Problem statement

Consider a rigid body moving in free space. Assume any inertial reference frame {F} fixed in space and a frame {M} fixed to the body. At each instance, the configuration (position and orientation) of the rigid body is uniquely described by a rigid body displacement from frame {F} to frame {M}. The set of all such displacements forms the special Euclidean group

$SE(3)$. Interpolation of a rigid body motion thus corresponds to interpolation on $SE(3)$.

The basic interpolation problem on $SE(3)$ is:

Problem 2.1 *Given a sequence $\{A_i\}_{i=1}^n$ of elements of $SE(3)$ and a sequence of times $\{t_i\}_{i=1}^n$, where $t_1 \leq t_2 \leq \ldots \leq t_n$, find a curve $\gamma(t)$ such that $\gamma(t_i) = A_i$.*

The problem can be refined for specific applications. We study the case when the velocities of the rigid body at times t_1 and t_n are prescribed. Furthermore, we wish to impose certain smoothness requirements on the motion. In particular, we concentrate on motions that are continuous at the level of acceleration (C^2). Most of the results in the paper can be generalized in a straight forward way if higher level of continuity is required. We therefore focus on the following problem:

Problem 2.2 *Given a sequence $\{A_i\}_{i=1}^n$ of elements of $SE(3)$, a sequence of times $\{t_i\}_{i=1}^n$, where $t_1 \leq t_2 \leq \ldots \leq t_n$, and the instantaneous twists V_1 and V_2 ($V_1, V_2 \in se(3)$) at times t_1 and t_n, find a curve $\gamma(t) \in C^2[t_1, t_n]$ such that $\gamma(t_i) = A_i$.*

The affine nature of the space (in which the rigid body moves) and practical considerations suggest that the interpolating curve should have some additional properties: (1) the trajectories must be independent of the representation of the motion in space (parameterization of $SE(3)$); (2) the trajectories should be independent of the choice of the inertial reference frame {F} and the body fixed frame {M}; (3) the trajectories must have good performance for the intended use.

We will present two different solutions to Problem 2.2. The first method is similar to those in [7, 10, 16] in that it is based on a particular representation of $SE(3)$ and interpolation in the appropriate parameter space using Euclidean methods. However, we choose a representation that allows easy geometric analysis of the computed trajectories and base our method on this geometric analysis, thereby guaranteeing its independence of the chosen representation of $SE(3)$. We show that the method satisfies (1) and (2) from the above, but it is difficult to quantify the performance of the solution. In the second part of the paper we therefore present a method that computes trajectories that optimize a chosen measure of smoothness. As before, by using differential geometry and basing the method on geometric analysis, we achieve its independence of the parameterization of the space. The method therefore satisfies (1) and (3). We show that (2) can be achieved by appropriate choice of Riemannian (or pseudo-Riemannian) metric on $SE(3)$. The drawback of this approach is that it is computationally expensive. There is therefore a trade-off between the computational efficiency and the optimality of the generated trajectories.

3 Interpolation based on the exponential map

We sketch a method that produces trajectories that do not depend on the parameterization of $SE(3)$, are independent on the choice of the inertial and body-fixed frame and have the required smoothness properties. However, the trajectories do not optimize a geometrically meaningful cost functional[1]. The main idea is to use properties of the exponential map on $SE(3)$ [13] to achieve the invariance of the trajectories with respect to the choice of the body-fixed and inertial reference frames.

Since rigid body displacement can be described with a homogeneous transformation matrix, $SE(3)$ can be also viewed as a set of all such matrices. A transformation matrix A can be uniquely expressed in the following way:

$$A = e^{\xi^1 L_1 + \xi^2 L_2 + \xi^3 L_3 + \xi^4 L_4 + \xi^5 L_5 + \xi^6 L_6} \tag{1}$$

where L_i is the basis of the Lie algebra $se(3)$ corresponding to the basis twists [21]. This map is a diffeomorphism in some neighborhood of the identity, so the vector $\xi = \{\xi_1, \ldots, \xi_6\}$ s defines local coordinates for $SE(3)$, also known as the canonical coordinates of the first kind [1]. Physically, vector ξ represents a twist [21]. In the rest of the paper we will also use the matrix representation of the twist and write $\text{vec}(\Xi) = \xi$ when Ξ is a matrix representation of ξ.

Let $\{A_i\}_{i=1}^n$ and $\{t_i\}_{i=1}^n$ be the sequences in Problem 2.2. We will look for an interpolating curve of the form:

$$\gamma(t) = A_1 \bar{\gamma}(t) \tag{2}$$

In this way, the interpolation problem becomes to find a curve $\bar{\gamma}(t)$ that interpolates the sequence $\{A_1^{-1} A_i\}_{i=1}^n$. Next, let the position $A_1^{-1} A_i$ be represented with a 6×1 vector ξ_i according to Eq. (1). Using interpolation in $\mathbb{R}^6$, it is easy to find a curve $\xi(t)$ such that $\xi(t_i) = \xi_i$. Obviously, the curve $\gamma(t) = A_1 e^{\Xi(t)}$, where $\text{vec}(\Xi) = \xi$, will pass through the point A_i at the time t_i and it is a suitable candidate for the interpolation. If we also wish to satisfy the boundary conditions on the velocities at t_1 and t_n, this poses additional requirements for the curve $\xi(t)$. If $\text{vec}(S)$ denotes the vector corresponding to an element $S \in se(3)$, the time derivative of the exponential map can be written in the form:

$$\text{vec}\left(\gamma(t)^{-1} \frac{d}{dt} \gamma(t)\right) = \Phi(\xi(t)) \dot{\xi}(t). \tag{3}$$

The 6×6 matrix Φ can be computed from the formula for the exponential map. The explicit expressions for

[1]They can be shown to minimize a cost functional formulated in the parameter space, but this cost functional does not have any geometric meaning.

the derivative of the exponential map on $SE(3)$ can be found in [2].

The instantaneous twist, V, representing the velocity of the rigid body, is given by [13]:

$$V = \gamma(t)^{-1} \frac{d}{dt} \gamma(t). \tag{4}$$

Hence, the expression for the velocity, expressed in the local coordinates, is $\dot{\xi} = \Phi(\xi)^{-1} \text{vec}(V)$. Let V_1 and V_2 denote the instantaneous twists at t_1 and t_n. To satisfy the boundary conditions on the velocities, the curve $\xi(t)$ must therefore satisfy the following equations:

$$\dot{\xi}(t)\Big|_{t_1} = \Phi(\xi_1)^{-1}\text{vec}(V_1) \qquad \dot{\xi}(t)\Big|_{t_n} = \Phi(\xi_n)^{-1}\text{vec}(V_n)$$

By computing a curve $\xi(t)$ (in the Euclidean space) that satisfies $\xi(t_i) = \xi_i$ and the above equations, we therefore obtain a curve that solves the interpolation problem. We now investigate properties of the trajectories generated in this way.

Smoothness Since the exponential map is smooth, the interpolating curve $\gamma(t)$ will be as smooth as the curve $\xi(t)$. We can therefore control the smoothness of the interpolating curve γ directly by controlling the smoothness of the curve ξ in the Euclidean space $\mathbb{R}^6$.

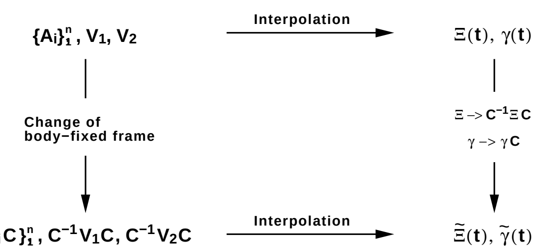
Left invariance The curve of the form (2) will always be left invariant, that is, invariant with respect to the choice of the inertial reference frame {F}. To see this, let us displace {F} so that its new position with respect to its initial position is given by the homogeneous transformation matrix C. Now the set of points that have to be interpolated becomes $\{CA_i\}_{i=1}^n$. But given the form (2), this means that the curve $\bar{\gamma}$ in Eq. (2) has to interpolate between the points $(CA_1)^{-1}CA_i = A_1^{-1}A_i$, which are the same as before. In addition, the velocity of the rigid body in the new frame is $(CA)^{-1}C\dot{A} = A^{-1}\dot{A}$, again the same as before. We conclude that if in the old frame the curve was $\gamma(t) = A_1\, e^{\Xi(t)}$, the curve in the new frame will simply be $\widetilde{\gamma}(t) = CA_1\, e^{\Xi(t)} = C\gamma(t)$, which proves left invariance. A similar proof was presented in [16].

Right invariance Consider again Problem 2.2 and the interpolating curve $\gamma(t) = A_1 e^{\Xi(t)}$. Now displace the body-fixed frame {M} through a displacement C. The interpolation points become $\{A_iC\}_{i=1}^n$, and the instantaneous twist in the new frame, denoted by $\tilde{V}$ is related to the twist in the original frame by the adjoint map [5, 11], $\tilde{V} = Ad_{C^{-1}}V = C^{-1}VC$. Consider a new coordinate curve $\tilde{\xi}(t) = \text{vec}(\Xi(t))$, given by $\tilde{\Xi}(t) = C^{-1}\Xi(t)C$. It is a property of the exponential map [5] that for any element $S \in se(3)$, $e^{C^{-1}SC} = C^{-1}e^{S}C$. Now take the curve $\widetilde{\gamma}(t) = \widetilde{A}_1\, e^{\widetilde{\Xi}(t)}$, where $\widetilde{A}_1 = A_1C$. A short derivation shows:

$$\begin{aligned} \widetilde{\gamma}(t) \;=\; & \widetilde{A}_1\, e^{\widetilde{\Xi}(t)} = A_1C\, e^{C^{-1}\Xi(t)C} = \\ & A_1C(C^{-1}\, e^{\Xi(t)}C) = A_1\, e^{\Xi(t)}C = \gamma(t)C. \end{aligned} \tag{5}$$

Since $\gamma(t)$ interpolates the points $\{A_i\}_{i=1}^n$, the curve $\widetilde{\gamma}(t)$ will interpolate the points $\{A_iC\}_{i=1}^n$ and it will satisfy the new boundary conditions $\widetilde{V}_i = C^{-1}V_iC$, $i = 1, 2$.

In order to show that the interpolation scheme is right invariant, we need to show that the curve $\widetilde{\xi}(t)$ is exactly the curve that would be produced by running the interpolation scheme in the transformed coordinate system. In other words, we need to show that the following diagram commutes:



In general this will not be true and we need to know how the interpolation between the coordinates ξ_i is performed. Obviously the interpolation scheme will be right invariant if and only if the set of curves $\xi(t) = \text{vec}(\Xi(t))$ that are generated by the scheme is invariant with respect to the transformation:

$$\text{Ad}_{C^{-1}} : \Xi(t) \mapsto C^{-1}\Xi(t)C, \tag{6}$$

for an arbitrary $C \in SE(3)$. Examples of families of curves that have this property and will therefore lead to a right invariant interpolation scheme are cubic splines and polynomials of a fixed degree which is greater or equal to $n-1$. The transformation in (6) is linear and therefore maps a cubic polynomial to a cubic polynomial. It is also not difficult to see that a cubic spline Ξ that solves the interpolation problem 2.2 formulated in the space of the coordinates of the first kind is unique. This implies that the diagram above must commute. A similar argument holds for the interpolation with polynomials of degree $n-1$: there is a unique polynomial of degree $n-1$ that will interpolate between n points. The diagram will also commute for the polynomials of higher degree, although in this case the interpolating curve is not unique.

Independence of the parameterization At the first sight, it seems that the proposed interpolation scheme is based on the particular choice of coordinates, that is, canonical coordinates of the first kind. However, while this choice of coordinates is convenient for the analysis, the actual interpolation does not necessarily have to be based on this representation. More precisely, the above analysis shows that we only need to establish the invariance of the family of the curves used for interpolation with respect to the map defined in (6).

4 Smooth interpolating curves

The interpolation scheme presented in the previous section yields smooth trajectories and is bi-invariant if the family of the interpolating curves in the parameter space is properly chosen. However, to prevent fast changes in the velocities or their higher derivatives, it is often desired to impose stricter smoothness requirements. More precisely, we can formulate a cost functional measuring the smoothness of a curve (the choice of the cost functional in general depends on the task) and find a curve that minimizes this cost functional.

It is well known [6] that in a Euclidean space cubic splines are the minima of the functional $J = \int_a^b \|\ddot{\gamma}\|^2 dt$. Noakes *et al.* [14] used this property to define cubic splines on an arbitrary manifold as the curves $A(t)$ minimizing $L_a = \int_a^b < \nabla_V V, \nabla_V V > dt$. In the function under the integral, $< ., . >$ stands for the Riemannian metric, ∇ is the corresponding Riemannian connection and $V = \frac{dA}{dt}$ (see [21]). In [14], necessary conditions are derived for such curves and result in the following equation

$$\nabla_V \nabla_V \nabla_V V + R(V, \nabla_V V)V = 0 \tag{7}$$

where $R(.,.)$ is the curvature.

There is no natural choice of metric for $SE(3)$ [13]. Metrics that do not depend on the choice of the reference frame {F} can be defined through a quadratic form on $se(3)$ [21]. In this paper we consider two choices. The first is the left invariant metric that was proposed by Park and Brockett [15]. The quadratic form on $se(3)$ for this metric is given by:

$$W = \begin{bmatrix} \alpha I & 0 \\ 0 & \beta I \end{bmatrix} \tag{8}$$

where α and β are arbitrary constants. An attractive property of this metric is that it is a product metric [20], which means that the rotational and the translational component of motion can be treated separately. This is also the metric induced by the kinetic energy of a spherical mass, a generalization of the concept of a point mass.

Another interesting metric is the Klein form [11]. It is defined thorough a quadratic form given by:

$$W = \begin{bmatrix} 0_{3\times3} & I_{3\times3} \\ I_{3\times3} & 0_{3\times3} \end{bmatrix} \tag{9}$$

The pseudo-Riemannian metric resulting from (9) has the important property that it is bi-invariant [22]. This means that the curves computed from (7) using this metric will be left and right invariant. Another interesting property of the metric is that the geodesics (shortest distance curve) are screw motions [22].

Having defined a metric, we can explicitly write the differential equations that result from Eq. (7). As a result we get a two-point boundary value problem that in general has to be solved numerically. The finite-difference method proves to be efficient for solving such boundary-value problems. The basic idea of the method is to discretize the differential equations in time: at each point, the system of differential equations is converted into a system of algebraic equations by approximating derivatives with finite differences. In this way, a system of coupled algebraic equations is obtained. The number of equations is usually fairly large. For detailed description of the method we refer the reader to [18] and the references therein.

To apply the finite difference method to Eq. (7), this higher order differential equation must be converted into a system of first-order differential equations. To do this, we have to define a set of local coordinates and compute the solution in the coordinates. One possible choice are the canonical coordinates of the first kind that were used in Section 3. We define:

$$x = \left[x_1^T x_2^T x_3^T x_4^T\right]^T ,$$

where

$$\begin{array}{ll} x_1 = \xi & x_3 = \dot{x}_2 \\ x_2 = \left[\omega^T v^T\right]^T & x_4 = \dot{x}_3. \end{array} \tag{10}$$

With these definitions and the relation given by Eq. (3), we can obtain the system of equations in the desired form.

Because of the requirement that the trajectory must pass through the intermediate points, we will have additional constraints of the form $x_1(t_i) = \xi_i$. The constraints can be simply handled with the finite-difference method since all we have to do at such points is replace Eq. (7) that gives the expression for $\dot{x}_4$ with the constraints, no additional computation is necessary. The boundary conditions at t_1 and t_n are also easily handled. They are given by:

$$\begin{array}{ll} x_1(t_1) = \xi_1 & x_2(t_1) = \mathrm{vec}(V_1) \\ x_1(t_n) = \xi_n & x_2(t_n) = \mathrm{vec}(V_2) \end{array} \tag{11}$$

Remark 4.1 Since we based the numerical method on computing the trajectories on which the first variation of the cost functional vanishes (first order optimality conditions), it is necessary to use additional tests to establish that the computed trajectory is a minimum. While this can be done relatively easily, it is in general much more difficult to prove that the

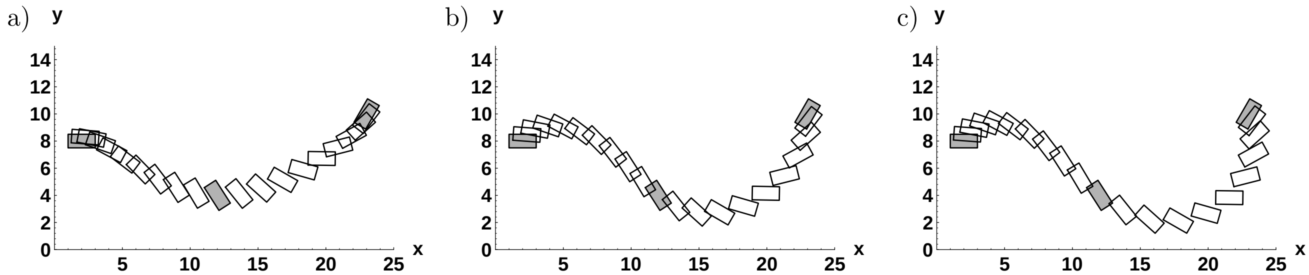


Figure 1: (a) Maximally smooth interpolating trajectory computed from the metric (8); (b) Spline interpolation in the coordinates of the first kind; (c) maximally smooth interpolation trajectory computed from the metric (9).

obtained minimum is a global minimum. For the problem at hand, we can conclude this with some certainty by analyzing the cases for which the global minimum can be analytically obtained and extrapolating the results to the case when the solution is computed numerically.

5 Examples

In this section we compare the interpolating trajectories computed with three different methods (a) the bi-invariant trajectories based on the exponential map and Euclidean interpolation in the space of canonical coordinates of the first kind; (b) the left-invariant trajectories obtained through minimization of the norm of the acceleration defined through the left-invariant metric (8); (c) the bi-invariant trajectories obtained through minimization of the “norm” of the acceleration defined through the bi-invariant metric[2] (9).

In Figure 1, the trajectories of motion for a rectangle in the plane $z = 0$ are computed. The frame {M} is attached at the center of mass of the rectangle with the axes parallel to the sides of the rectangle. Let $P = \{\theta, x, y\}$ denote the orientation and position of the frame {M} in the plane and $V = \{\omega_z, v_x, v_y\}$ the corresponding velocity. The interpolation problem was given by:

$$\begin{array}{lll} P_1 = \{0, 2, 8\} & t_1 = 0 & \\ P_2 = \{-1, 12, 4\} & t_2 = \frac{1}{2} & V_1 = \{-1, 3, 10\} \\ P_3 = \{\frac{\pi}{3}, 23, 10\} & t_3 = 1 & V_2 = \{2, 2, 5\} \end{array}$$

In the figure, the three prescribed positions are represented by a shaded rectangle. Figure 1.a shows a trajectory composed of geodesics (with the metric (8)) between the consecutive positions. Obviously, this trajectory is not smooth at the intermediate point. Furthermore, it does not satisfy the boundary conditions on the velocity. Figure 1.b shows the maximally smooth trajectory computed from Eq. (7) using the metric (8). The translational part of the motion in this case is given by the cubic spline in $\mathbb{R}^2$. Since the metric used to find this motion is left invariant but not bi-invariant, the motion is independent of the choice of the inertial reference frame {F}, but it does depend on the choice of the body-fixed reference frame {M}. The next figure, 1.c, shows the trajectory computed by interpolation in the canonical coordinates and with the use of exponential map. As shown in Section 3, this trajectory is bi-invariant. Finally, the trajectory shown in Figure 1.d is the maximally smooth trajectory computed from Eq. (7) using the metric (9). Since the metric is bi-invariant, so are the maximally smooth trajectories. The last two trajectories are similar, but they are not the same. The figures suggest, and it can be shown rigorously, that in the planar case the rotational parts of the motions obtained by any of the above schemes are the same. This is not true for a general spatial case.

[2]Since the metric is not positive-definite, the norm can be negative. See [20] for a discussion.

6 Concluding remarks

We discussed different methods for interpolation between a given set of positions and orientations. A typical approach to the problem is to interpolate the orientations and the positions separately and then combine the resulting interpolating curves. We show that by viewing the problem as an interpolation problem on $SE(3)$, we can investigate several properties of the trajectories. In particular, we can systematically analyze the invariance of the interpolating curves with respect to the choice of the inertial and the body-fixed frames and the smoothness properties of the trajectories.

We proposed two different interpolation schemes. In the first, the given set of elements of $SE(3)$ is represented by the canonical coordinates of the first kind and interpolation with cubic splines is performed on these coordinates by treating them as vectors in the Euclidean space. Properties of the exponential map imply that the resulting trajectories are bi-invariant. In the second scheme, the methods of variational calculus on $SE(3)$ are used to compute the maximally smooth trajectories according to a suitable measure of smoothness. With this method we computed the trajectories that are a generalization of cubic splines from the Euclidean case. The advantage of the first scheme is that the trajectories are easy to compute. The second scheme is attractive since it produces the “best possible” interpolating trajectories, but the solution must be computed numerically. In this paper, the trajectories are obtained by solving a two-point boundary value problem with a finite-difference method.

We computed the interpolating trajectories for a case when the motion takes place in a plane and for a spatial case. In both cases, the maximally smooth motions computed from the metric (8) appear to be the most natural. However, these trajectories must be computed numerically and they depend on the choice of the body-fixed reference frame. The examples also suggest that these motions might be well approximated with the trajectories that are obtained by separate interpolation on $SO(3)$ and $\mathbb{R}^3$ with the first method. An interesting direction for future work would be to compute the estimates on the difference between trajectories obtained in this way and the optimal trajectories.

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