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# Affine Connections for the Cartesian Stiffness Matrix

Miloš Žefran and Vijay Kumar
General Robotics and Active Sensory Perception (GRASP) Laboratory, University of Pennsylvania
3401 Walnut Street, Room 301C, Philadelphia, PA 19104-6228

## Abstract

In this paper, we study the $6 \times 6$ Cartesian stiffness matrix. We show that the stiffness of a rigid body subjected to conservative forces and moments is described by a $(0,2)$ tensor which is the Hessian of the potential function. The key observation of the paper is that since the Hessian depends on the choice of an affine connection in the task space, so will the Cartesian stiffness matrix. Further, the symmetry of the Hessian and thus of the stiffness matrix depends on the symmetry of the connection. The connection that is implicit in the definition of the Cartesian stiffness matrix through the joint stiffness matrix [1] is made explicit and shown to be symmetric. In contrast, the direct definition of the Cartesian stiffness matrix in [2, 3, 4] is shown to be derived from an asymmetric connection. A numerical example is provided to illustrate the main ideas of the paper.

## 1 Introduction

This paper considers the static analysis of conservative systems in which the associated potential energy $\Phi$ is a function of position only. Specifically, we consider a rigid body attached to the ground through one or more articulations with joints that are flexible or under compliance control.

In $\mathbb{R}^n$, the stiffness matrix $\mathcal{K}$ consists of the second partial derivatives of the potential energy with respect to the coordinates:

$$\mathcal{K}_{ij} = \frac{\partial^2 \Phi}{\partial q_i \partial q_j}. \tag{1}$$

A position and orientation of a rigid body can be described with a rigid body displacement. The set of all such displacements is known in robotics as the task space. This space is not Euclidean so Eq. (1) can not be used to compute the stiffness matrix. The velocities in the task space are described in terms of a six dimensional basis consisting of zero pitch twists (pure rotations) along the axes of the reference frame and infinite pitch twists (pure translations) parallel to the axes of the reference frame. The forces and moments can also be expressed as components with respect to a similar basis of wrenches. The changes in forces and moments with respect to small motions along the basis twists are given by a $6 \times 6$ Cartesian stiffness matrix.

The main objective of the paper is to establish a precise definition of the Cartesian stiffness matrix. The set of spatial rigid body displacements forms a six-dimensional differentiable manifold. A position and orientation of a rigid body is a point on this manifold. The vector representing the generalized velocity of the rigid body is a tangent vector at that point, while the generalized force can be thought of as a cotangent vector. The stiffness matrix requires differentiation of the generalized force in the direction of the generalized velocity. Roughly speaking, differentiation requires comparison of tangent (cotangent) vectors at two nearby, but *different* points. To formalize this process, we need to endow the manifold with an *affine connection*. In this paper, we will make explicit the connections that are implicitly used in previous work and make sense for applications in robotics.

In the robotics literature, the $6 \times 6$ Cartesian stiffness matrix is generally defined through the Hessian computed in the joint space. The force displacement equations in the joint coordinates define a joint stiffness matrix. This stiffness matrix can be transformed to a Cartesian stiffness matrix with the manipulator Jacobian. More precisely, if the map between the joint space velocity $\dot{q}$ and the task space velocity $\{\omega^T, v^T\}^T$ is $\{\omega^T, v^T\}^T = J\dot{q}$, the Cartesian stiffness matrix is [1]:

$$K = J^{-T} \mathcal{K} J^{-1} \tag{2}$$

We will show that this definition of the Cartesian stiffness matrix implicitly defines an affine connection for the task space. However, the connection clearly depends on the physical characteristics of the robot manipulator and is therefore not intrinsic to the task space. In other words, for the same potential function on the task space, a different manipulator would have a different Cartesian stiffness matrix.

Griffis and Duffy [2] define the Cartesian stiffness matrix directly in the task space using a basis for twists and wrenches. Such definition is independent of the coupling between the end effector and the base, however, the resulting stiffness matrix is no longer symmetric. A similar definition is also used by Ciblak and Lipkin [3] and Howard *et al.* [4]. We will show that these papers also implicitly assume an affine connection and it is the properties of the connection that cause the asymmetry of the stiffness matrix.

The outline of the paper is as follows. In the next section we introduce basic ideas in differential geometry and Lie groups in the context of the kinematics of the motion of a rigid body. Section 3 is concerned with the definition of the covariant derivative and the Cartesian stiffness matrix. In Section 4, we show that various definitions of the Cartesian stiffness matrices in the literature correspond to different choices of the affine connection in the task space. A numerical example taken from [2] is presented in Section 5 to illustrate this fact. Section 6 contains some concluding remarks.

## 2 Geometry and kinematics

Consider a rigid body moving in free space. Assume any inertial reference frame {F} fixed in space and a frame {M} fixed to the body as shown in Figure 1. At each instance, the configuration (position and orientation) of the rigid body can be described by a homogeneous transformation matrix corresponding to the displacement from frame {F} to frame {M}. These transformations form a Lie group $SE(3)$, the special Euclidean group in three-dimensions [5].

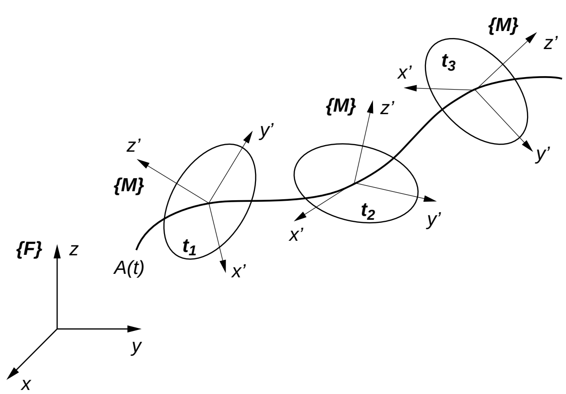


Figure 1: The inertial (fixed) frame and the moving frame attached to the rigid body.

On a Lie group, the tangent space at the group identity has the structure of a Lie algebra. The Lie algebra of $SE(3)$ is denoted by $se(3)$ and is given by:

$$se(3) = \left\{ \begin{bmatrix} \Omega & v \\ 0 & 0 \end{bmatrix}, \Omega \in \mathbb{R}^{3\times 3}, v \in \mathbb{R}^3, \Omega^T = -\Omega \right\}.$$

A $3 \times 3$ skew-symmetric matrix $\Omega$ can be uniquely identified with a vector $\omega \in \mathbb{R}^3$ so that for an arbitrary vector $x \in \mathbb{R}^3$, $\Omega x = \omega \times x$, where $\times$ is the cross product in $\mathbb{R}^3$. Each element $T \in se(3)$ can be thus identified with a vector pair $\{\omega, v\}$.

Since $se(3)$ is a vector space, any element can be expressed as a $6 \times 1$ vector of components corresponding to a chosen basis. The standard basis that will be used throughout the paper is:

$$L_1 = \begin{bmatrix} 0&0&0&0 \\ 0&0&-1&0 \\ 0&1&0&0 \\ 0&0&0&0 \end{bmatrix} \; L_2 = \begin{bmatrix} 0&0&1&0 \\ 0&0&0&0 \\ -1&0&0&0 \\ 0&0&0&0 \end{bmatrix} \; L_3 = \begin{bmatrix} 0&-1&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}$$
$$L_4 = \begin{bmatrix} 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix} \; L_5 = \begin{bmatrix} 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix} \; L_6 = \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{bmatrix} \tag{3}$$

This basis has the property that the components of an element $T \in se(3)$ are given precisely by the vector pair $\{\omega, v\}$ mentioned above.

The product operation on a Lie algebra is called a Lie bracket. The Lie bracket of two elements $T_1, T_2 \in se(3)$ is defined by:

$$[T_1, T_2] = T_1 T_2 - T_2 T_1. \tag{4}$$

### 2.1 Velocity of a rigid body

Consider a rigid body moving in space. The motion of the rigid body can be described by a curve $A(t) : \mathbb{R} \to SE(3)$. The tangent vector to this curve, $\frac{dA}{dt}$, is the velocity of the rigid body and can be mapped to an element $T(t)$ of the Lie algebra $se(3)$ by:

$$T(t) = A(t)^{-1}\dot{A}(t) = \begin{bmatrix} R^T\dot{R} & -R^T\dot{d} \\ 0 & 0 \end{bmatrix}. \tag{5}$$

If $T(t) = \{\omega(t), v(t)\}$ is expressed in the basis (3), then $\omega(t)$ is the angular velocity of the rigid body while $v(t)$ is the linear velocity of the origin of the frame {M}, both expressed in the body-fixed frame {M}. Therefore, $T(t)$ is the *instantaneous twist* [6] in the body-fixed frame and the Lie algebra $se(3)$ is isomorphic to the set of all twists [5].

### 2.2 Left invariant vector fields

A differentiable vector field is a smooth assignment of a tangent vector to each point on the manifold. At each point, a vector field defines a unique *integral curve* to which it is tangent [7]. Formally, a vector field $X$ is a derivation operator which, given a real-valued differentiable function $f$, returns its derivative along the integral curves of $X$.

In this paper, we will be particularly interested in the *left invariant vector fields* on $SE(3)$. From any twist $T \in se(3)$ we can generate a left invariant vector field, $\hat{T}$, by assigning to $A \in SE(3)$ a vector:

$$\hat{T}(A) = AT, \tag{6}$$

Since the vectors $L_1, L_2, \ldots, L_6$ are a basis for the Lie algebra $se(3)$, the vectors $\hat{L}_1(A), \ldots, \hat{L}_6(A)$ form a basis of the tangent space at any point $A \in SE(3)$. Therefore, *any* vector field $X$ can be expressed as a linear combination of the left invariant vector fields:

$$X = \sum_{i=1}^{6} X^i \hat{L}_i \tag{7}$$

where the coefficients $X^i$ are real-valued functions. Equation (5) shows that:

$$\dot{A} = A\left(\sum_{i=1}^{6} T^i L_i\right) = \sum_{i=1}^{6} T^i \hat{L}_i(A). \qquad (8)$$

We conclude that if the velocity of the rigid body, $\dot{A}$, is expressed in the basis $\hat{L}_1, \ldots, \hat{L}_6$, its components are equal to the Cartesian components of the instantaneous twist:

$$\omega = [T^1, T^2, T^3]^T, \qquad v = [T^4, T^5, T^6]^T.$$

We will therefore refer to the elements $\hat{L}_1, \ldots, \hat{L}_6$ of the basis for the left invariant vector fields as *basis twists.*

### 2.3 Twists and wrenches

If a force $F$ and a moment $\tau$ act on a rigid body, we refer to the vector pair $W = \{\tau, F\}$ as a *wrench.* When a wrench $W$ acts on a rigid body that undergoes a twist $T = \{\omega, v\}$ over a time interval $\Delta t$, it produces the work:

$$\Delta E = (F^T v + \tau^T \omega)\Delta t,$$

which is a scalar. Wrenches therefore belong to the dual of the vector space of twists, $se^*(3)$. The instantaneous twist is the Lie algebra element that corresponds to the current value of the velocity vector field. By analogy, the force[1] physically corresponds to the dual of the vector field (a one-form) [8] and the wrench is obtained by mapping this one-form to an element in the dual of the Lie algebra.

Given a basis for the vector fields $\{\hat{L}_i\}$, there exists a natural basis for one-forms, $\{\hat{\lambda}^i\}$, called *the dual basis*, which satisfies $<\hat{\lambda}^i\,;\,\hat{L}_j>= \delta^i_j$, where $<\hat{\lambda}^i\,;\,\hat{L}_j>$ represents the action of the one-form $\hat{\lambda}^i$ on the vector field $\hat{L}_j$ and $\delta^i_j$ is the Kronecker $\delta$. It is not difficult to see that the components of the force one-form in the basis $\{\hat{\lambda}^i\}$ that is dual to the basis twists $\{\hat{L}_i\}$, are exactly the components of the corresponding wrench in $se^*(3)$.

### 2.4 Forces in a potential field

In $\mathbb{R}^3$, the force generated by a potential field $\phi$ is equal to the negative gradient of the potential field, $F = -\text{grad}(\phi)$. This can be generalized to an arbitrary manifold if we generalize the notion of gradient [9]. A gradient of a real-valued function $f$, denoted by $df$, is a one-form, whose action on an arbitrary vector field $X$ is defined by:

$$<df\,;\,X>= X(f), \qquad (9)$$

where $X(f)$ is the derivative of $f$ along the integral curves of $X$ (see Section 2.2). The force one-form $\mathcal{F}$ corresponding to a potential field $\Phi$ is therefore:

$$\mathcal{F} = -d\Phi. \qquad (10)$$

To obtain the wrench that corresponds to the force one-form $\mathcal{F}$ at a point $A \in SE(3)$, the one form must be expressed in the basis dual to the basis twists. The components of the wrench are therefore given by $W_i =<\mathcal{F}\,;\,\hat{L}_i>= -\hat{L}_i(\Phi)$ and $\mathcal{F} = W_i\,\hat{\lambda}^i$. Here and in the rest of the paper we use the Einstein summation convention.

## 3 Cartesian Stiffness Matrix

### 3.1 Covariant derivative

Differentiation of vector fields and one forms on the manifold is not defined unless the manifold is endowed with *an affine connection.* Given a curve $A(t)$ on the manifold, the affine connection specifies how an element $X$ of the tangent space at a point $A(t_1)$ can be mapped to an element $X'$ of the tangent space at some other point $A(t_2)$. The vector $X'$ is called the parallel transport of $X$ along $A(t)$. In this way, we can define a *covariant derivative* of a vector field $X(t)$ along a curve $A(t)$ by:

$$\left.\frac{DX}{dt}\right|_{t_0} = \lim_{h\to 0} \frac{X^{t_0}(t_0+h) - X(t_0)}{t}, \qquad (11)$$

where $X^{t_0}(t_0+h)$ is the parallel transport of the vector $X(A(t_0+h))$ along $A(t)$ to the point $A(t_0)$. By taking the covariant derivative of a vector field $X$ along the integral curve of another vector field $Y$, we obtain the covariant derivative, $\nabla_Y X$, of the vector field $X$ with respect to the vector field $Y$:

$$\nabla_Y X|_{A_0} = \left.\frac{DX}{dt}\right|_{t_0}, \qquad (12)$$

where $\frac{DX}{dt}$ is taken along the integral curve of $Y$ passing through $A_0$ at $t = t_0$. If $\{X_i\}$ is a set of basis vector fields, the coefficients $\Gamma^k_{ji}$ of covariant derivative of a basis vector field along another basis vector field:

$$\nabla_{X_i} X_j = \Gamma^k_{ji} X_k, \qquad (13)$$

are called *Christoffel symbols*[2].

**Definition 3.1** *If for all vector fields $X$ and $Y$, a connection satisfies:*

$$\nabla_X Y - \nabla_Y X = [X, Y], \qquad (14)$$

*the connection is said to be* symmetric.

A covariant derivative of a one-form can be defined through the covariant derivative of a vector field [9]:

$$<\nabla_U \mathcal{F}\,;\,V>= U(<\mathcal{F}\,;\,V>)- <\mathcal{F}\,;\,\nabla_U V>. \qquad (15)$$

Loosely speaking, this is a "generalization" of the Leibniz' rule.

[1] By force we mean the generalized force consisting of both, forces and torques acting on the rigid body.

[2] Some texts [7] reserve the term only for the coordinate basis vectors. We follow the more general definition from [9] in which the basis vectors can be arbitrary.

### 3.2 Formal definition of the Cartesian stiffness matrix

The Cartesian stiffness matrix is obtained by differentiating the force one-form in the directions of the basis twists which means that we need to compute the derivatives $\nabla_{\hat{L}_i} d\Phi$. Since we want to know how the *wrench components* of the force one-form change, the one-forms $\nabla_{\hat{L}_i} d\Phi$ must be expressed in the basis dual to the basis twists. This leads to the following definition for the coefficients of the stiffness matrix:

$$K_{ij} = < \nabla_{\hat{L}_j} d\Phi \, ; \, \hat{L}_i > . \tag{16}$$

(The minus sign is omitted to conform to the usual definition of the stiffness matrix in the literature.) The operator $\nabla d\Phi$, defined on an arbitrary pair of vector fields $X$ and $Y$ by:

$$\nabla d\Phi(X, Y) = < \nabla_X d\Phi \, ; \, Y >, \tag{17}$$

is known as the *Hessian* of the function $\Phi$ and can be shown to be a $\binom{0}{2}$ tensor [10]. Equation (16) therefore implies:

**Proposition 3.1** *A component $K_{ij}$ of the Cartesian stiffness matrix is obtained by evaluating the Hessian $\nabla d\Phi$ on the pair of basis twists $\hat{L}_i$ and $\hat{L}_j$.*

An important consequence is that the Cartesian stiffness matrix does not depend on the parameterization of the task space [11].

Equation (16) can be expanded using Eq. (15):

$$\begin{aligned} K_{ij} &= \hat{L}_j(< d\Phi \, ; \, \hat{L}_i >) - < d\Phi \, ; \, \nabla_{\hat{L}_j} \hat{L}_i > \\ &= (\hat{L}_j \hat{L}_i - \nabla_{\hat{L}_j} \hat{L}_i)(\Phi). \end{aligned} \tag{18}$$

This equation shows that *the stiffness matrix depends on the choice of the affine connection.* The properties of the affine connection will thus affect the Hessian and consequently the stiffness matrix. It is not difficult to show (see [11]):

**Proposition 3.2** *If the affine connection is symmetric, then the Hessian is symmetric.*

**Corollary 3.3** *If the connection used to compute the Cartesian stiffness matrix is symmetric, the Cartesian stiffness matrix itself is symmetric.*

## 4 Definitions of the stiffness matrix in the literature

### 4.1 Cartesian stiffness matrix obtained from the joint stiffness matrix

Assume we have a non-redundant manipulator. Let $q = \{q_1, \ldots, q_6\}^T$ be the vector of joint coordinates and assume that the robot does not have a singularity at $q = \{q_1^0, \ldots, q_6^0\}^T$. Choose an end-effector frame {M} which is fixed to the last link of the manipulator and set the inertial reference frame {F} to be the position of the end-effector frame at $q = \{q_1^0, \ldots, q_6^0\}^T$ (see Fig. 1). Point $q = \{q_1^0, \ldots, q_6^0\}^T$ therefore corresponds to the identity element of $SE(3)$ and in some neighborhood $U$, $SE(3)$ is (locally) parameterized by the joint coordinates.

At every point $A \in U$, we can choose the so called coordinate basis for the tangent space, which is given by $E_i = \frac{\partial}{\partial q_i}(.)$. Since any other vector field can be expressed as a linear combination of $E_i$'s, we can define the affine connection by setting:

$$\nabla_{E_i} E_j = 0, \qquad i, j = 1, \ldots, 6. \tag{19}$$

Note that for the coordinate basis all the Lie brackets vanish. We therefore have:

$$\nabla_{E_i} E_j - \nabla_{E_j} E_i = 0 = [E_i, E_j], \tag{20}$$

which means that the connection defined by (19) is symmetric.

The joint stiffness matrix $\mathcal{K}$ is obtained by evaluating the Hessian of the potential function $\Phi$ on the pairs of the coordinate basis vectors. The entries of the joint stiffness matrix are therefore given by:

$$\mathcal{K}_{ij} = (E_j E_i - \nabla_{E_j} E_i)\Phi. \tag{21}$$

Recalling the definition of the connection (19), this simplifies to:

$$\mathcal{K}_{ij} = E_j E_i(\Phi) = \frac{\partial^2 \Phi}{\partial q_j \partial q_i}. \tag{22}$$

We have therefore obtained the familiar expression for the joint stiffness matrix.

To find the Cartesian stiffness matrix that corresponds to the connection defined in (19) we must evaluate the Hessian $\nabla d\Phi$ on the pairs of basis twists. For this we need to express the basis twists in the canonical basis. Recall that the mapping between the basis twists and the canonical basis is given exactly by the Jacobian matrix (see [5]). In particular:

$$E_i = \gamma_i^j \hat{L}_j, \tag{23}$$

where $\gamma_i^j$ is the element $J_{ji}$ of the Jacobian matrix. Let $\alpha_i^j$ denote the element of the inverse Jacobian, $[\alpha_i^j] = [\gamma_i^j]^{-1}$. We can therefore write:

$$\hat{L}_i = \alpha_i^j E_j. \tag{24}$$

To compute the entry of the Cartesian stiffness matrix we use the expression (16):

$$\begin{aligned} K_{ij} &= < \nabla_{\hat{L}_j} d\Phi \, ; \, \hat{L}_i > = < \nabla_{(\alpha_j^l E_l)} d\Phi \, ; \, (\alpha_i^m E_m) > \\ &= \alpha_j^l \alpha_i^m < \nabla_{E_l} d\Phi \, ; \, E_m > . \end{aligned} \tag{25}$$

In this derivation we have used the linearity of the natural pairing $< . \, ; \, . >$ and the linearity of the affine connection in the first factor. Since $< \nabla_{E_l} d\Phi \, ; \, E_m > = \mathcal{K}_{lm}$, it is easy to see that Eq. (25) is exactly Eq. (2):

$$K = J^{-T} \mathcal{K} J^{-1}. \tag{26}$$

We can also compute the Christoffel symbols of the connection (19) for the basis twists:

$$\nabla_{\hat{L}_j}\hat{L}_i = \nabla_{(\alpha_j^l E_l)}(\alpha_i^m E_m) \quad (27)$$
$$= (\alpha_j^l E_l)(\alpha_i^m)E_m + \alpha_i^m \alpha_j^l \nabla_{E_l} E_m = \alpha_j^l \frac{\partial \alpha_i^m}{\partial q_l}\gamma_m^k \hat{L}_k,$$

where the last equality is obtained by using (23) and definition of the connection (19).

### 4.2 Cartesian stiffness matrix defined in [2, 3, 4]

In this section we compare the formal definition of the Cartesian stiffness matrix based on the Hessian of the potential function with the definition found in [2, 3, 4]. In these works, the coefficient of the stiffness matrix $K_{ij}$ is computed by taking a small displacement $\Delta T^j$ in the direction of the basis twist $\hat{L}_j$, computing the corresponding change of the wrench component $W_i$ and taking the limit of the quotient of the two as the displacement goes to 0:

$$K_{ij} = -\lim_{\Delta T^j \to 0} \frac{\Delta W_i}{\Delta T^j}. \quad (28)$$

We observe that this corresponds precisely to derivative of $W_i$ along the integral curve of $\hat{L}_j$:

$$K_{ij} = -\hat{L}_j(W_i). \quad (29)$$

But $W_i = -< d\Phi\,; \hat{L}_i >= -\hat{L}_i(\Phi)$, so the expression for the element $K_{ij}$ becomes:

$$K_{ij} = \hat{L}_j \hat{L}_i(\Phi). \quad (30)$$

If we compare Eq. (30) with Eq. (18), it is immediately apparent that the term with the connection is missing in (30). Hence, the definition of the stiffness matrix in [2, 3, 4] requires:

$$\nabla_{\hat{L}_i}\hat{L}_j = 0 \qquad \forall i, j. \quad (31)$$

If the vector fields are expressed with the basis twists $\{\hat{L}_i\}$, the Christoffel symbols for this connection all vanish:

$$\Gamma_{ji}^k = 0 \qquad \forall i, j, k. \quad (32)$$

Definition (31) implies that the connection is not symmetric:

$$0 = \nabla_{\hat{L}_i}\hat{L}_j - \nabla_{\hat{L}_j}\hat{L}_i \neq [\hat{L}_i, \hat{L}_j]. \quad (33)$$

The asymmetry of the connection results in the asymmetry of the stiffness matrix. Equation (30) yields:

$$K_{ij} - K_{ji} = [\hat{L}_j, \hat{L}_i](\Phi), \quad (34)$$

and since the basis twists are not the coordinate basis, the Lie brackets do not vanish. It is also easy to see that at stationary points of the potential field $\Phi$, the stiffness matrix becomes symmetric (at a stationary point, $X(\Phi) = 0$ for an arbitrary vector field $X$).

## 5 Example

We borrow the example from [2]. Take the parallel platform depicted on Fig. 2 and assume that the current configuration of the two platforms is:

$$O = [0.0\ \ 0.0\ \ 0.0]^T \qquad P = [7.0\ \ 0.0\ \ 0.0]^T$$
$$Q = [3.5\ \ 6.0\ \ 0.0]^T \qquad r = [10\ \ 4\ \ 12]^T$$
$$s = [14.0\ \ 8.0\ \ 16.0]^T \qquad t = [14.5\ \ 1.1\ \ 16.5]^T$$

The springs represent translational degrees of freedom so the joint coordinates $q_i$ can be chosen to be the lengths of the links connecting the base $\triangle OPQ$ with the platform (end-effector) $\triangle rst$. The potential energy for the linkage is given by:

$$\Phi = \sum_{i=1}^{6} \frac{1}{2} k_i (q_i - l_i)^2, \quad (35)$$

where $k_i$ are the spring constants and $l_i$ are the lengths of the springs in the unloaded configuration. For the example, we chose:

$$k_1 = 10N/cm, \quad k_2 = 20N/cm, \quad k_3 = 30N/cm,$$
$$k_4 = 40N/cm, \quad k_5 = 50N/cm, \quad k_6 = 60N/cm,$$

and

$$l_1 = 11cm, \quad l_2 = 12cm, \quad l_3 = 13cm,$$
$$l_4 = 14cm, \quad l_5 = 15cm, \quad l_6 = 16cm.$$

In a real manipulator, the spring constant $k_i$ would correspond to the equivalent stiffness of the $i$th actuator (combination of the stiffness of its transmission and the stiffness set by the control law). The unloaded spring lengths $l_i$ can be computed from the equation $\tau_i = -k_i(q_i - l_i)$, where $\tau_i$ is the force exerted by the $i$th actuator in the current configuration.

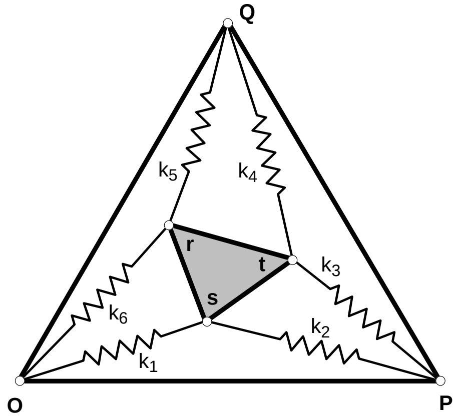


Figure 2: Top view of the parallel platform.

Equations (22) and (35) suggest that the joint stiffness matrix is $\mathcal{K} = \text{diag}\{10, 20, 30, 40, 50, 60\}$. We can also readily compute the value of the inverse Jacobian at the current configuration by computing changes in the link lengths as the end-effector moves in the directions of the basis twists. The resulting matrix is:

$$J^{-1} = \begin{bmatrix} 0.00 & 0.00 & 0.00 & 0.62 & 0.35 & 0.70 \\ 0.00 & -5.83 & 2.92 & 0.37 & 0.42 & 0.83 \\ 0.00 & -6.36 & 0.41 & 0.41 & 0.06 & 0.91 \\ 4.89 & -2.82 & -4.12 & 0.54 & -0.24 & 0.81 \\ 5.27 & -3.04 & -3.38 & 0.47 & -0.15 & 0.87 \\ 0.00 & 0.00 & 0.00 & 0.62 & 0.25 & 0.74 \end{bmatrix}.$$

We can now evaluate Eq. (25) to compute the Cartesian stiffness matrix obtained from the joint stiffness matrix:

$$K_j = \begin{bmatrix} 2345.9 & -1354.5 & -1695.2 & 229.3 & -87.1 & 387.0 \\ -1354.5 & 2674.9 & 559.7 & -253.8 & -9.6 & -493.8 \\ -1695.2 & 559.7 & 1423.5 & -141.6 & 90.5 & -219.8 \\ 229.3 & -253.8 & -141.6 & 57.3 & 6.4 & 87.2 \\ -87.1 & -9.6 & 90.5 & 6.4 & 12.0 & 7.7 \\ 387.0 & -493.8 & -219.8 & 87.2 & 7.7 & 140.7 \end{bmatrix}$$

To compute the Cartesian stiffness matrix given by Eq. (30), the potential energy (35) must be differentiated twice in the directions of the basis twists. The computation yields:

$$K_a = \begin{bmatrix} 2141.4 & -1725.1 & -3895.1 & 206.9 & -581.5 & 466.5 \\ -2094.7 & 4107.4 & -336.5 & 303.5 & 5.3 & -836.8 \\ -1518.7 & 608.6 & 3263.9 & -239.5 & 516.7 & -212.2 \\ 206.9 & -202.4 & -180.2 & 80.0 & 5.2 & 75.6 \\ -75.5 & 5.3 & 212.0 & 5.2 & 39.3 & 5.2 \\ 407.2 & -532.2 & -212.2 & 75.6 & 5.2 & 150.6 \end{bmatrix}$$

As expected, in this case the matrix is asymmetric since an asymmetric connection was used in the computation.

The computations show that different connections lead to completely different stiffness matrices for the *same physical system*. The proper choice of the connection depends on the application. The joint stiffness matrix is naturally found in the computations required for stiffness or hybrid control of manipulators. However, in tasks such as multifingered grasping, there is no natural choice of coordinates for the grasped object and the definition of the stiffness matrix in (30) is preferred [12]. Similarly, Eq. (30) is generally used when measuring the stiffness matrices for force sensors or RCC wrists [3]. The main point is that it is necessary to understand the role of the connection and its effect on the stiffness matrix when the system is subjected to an external load. This is especially critical when comparing stiffness matrices that are used in different algorithms.

## 6 Conclusion

We presented a coordinate-free formulation of the Cartesian stiffness matrix for conservative mechanical systems in which the potential field $\Phi$ is a function of position only. We showed that it is necessary to define an affine connection on $SE(3)$ in order to compute the stiffness matrix. We described two approaches that are used in robotics to define a stiffness matrix and that implicitly define an affine connection. The traditional approach of computing the stiffness matrix in the joint space and then transforming it to the Cartesian space [1] defines a connection on $SE(3)$ that depends on the manipulator and the parameterization of its joint space. In contrast, the more direct approach of [2, 3] implicitly assumes an affine connection on $SE(3)$ that is asymmetric. We presented a simple example to illustrate the differences in the two approaches.

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## References

[1] J. Salisbury, "Active stiffness control of a manipulator in Cartesian coordinates," in *Proceedings of the 19th IEEE Conference on Decision and Control*, Dec. 1980.

[2] M. Griffis and J. Duffy, "Global stiffness modeling of a class of simple compliant couplings," *Mechanism and Machine Theory*, vol. 28, no. 2, pp. 207–224, 1993.

[3] N. Ciblak and H. Lipkin, "Asymmetric Cartesian stiffness for the modeling of compliant robotic systems," in *Proc. of the 23rd Biennial ASME Mechanisms Conf.*, (Minneapolis, MN), 1994.

[4] W. S. Howard, M. Žefran, and V. Kumar, "On the 6x6 stiffness matrix for three dimensional motions," *Mechanism and Machine Theory*, 1996. In press.

[5] R. M. Murray, Z. Li, and S. S. Sastry, *A Mathematical Introduction to Robotic Manipulation.* CRC Press, 1994.

[6] J. M. McCarthy, *An Introduction to Theoretical Kinematics.* MIT Press, 1990.

[7] M. P. do Carmo, *Riemannian geometry.* Boston: Birkhauser, 1992.

[8] J. C. Simo, "The (symmetric) Hessian for geometrically nonlinear models in solid mechanics: intrinsic definition and geometric interpretation," *Computer Methods in Applied Mechanics and Engineering*, vol. 96, no. 2, pp. 189–200, 1992.

[9] B. F. Schutz, *Geometrical Methods of Mathematical Physics.* Cambridge: Cambridge University Press, 1980.

[10] B. O'Neill, *Semi-Riemannian Geometry.* New York: Academic Press, 1983.

[11] M. Žefran and V. Kumar, "Coordinate-free formulation of the cartesian stiffness matrix," in *Recent Advances in Robot Kinematics* (V. Parenti-Castelli and J. Lenarčič, eds.), Dordrecht: Kluwer Academic Publishers, 1996.

[12] W. S. Howard and V. Kumar, "On the stability of grasped objects," *IEEE Transactions on Robotics and Automation*, vol. 12, no. 6, pp. 904–917, 1996.