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Planning of smooth motions on $SE(3)$

Miloš Žefran and Vijay Kumar
General Robotics and Active Sensory Perception (GRASP) Laboratory, University of Pennsylvania
3401 Walnut Street, Suite 301C, Philadelphia, PA 19104-6228

Abstract

This paper addresses the general problem of generating smooth trajectories between an initial and a final position and orientation. A functional depending on velocity and its higher derivatives involving a left invariant Riemannian metric on $SE(3)$ is used to measure the smoothness of a trajectory. The problem of determining a smooth trajectory between two points is formulated as a variational problem on $SE(3)$. We derive necessary conditions for the shortest distance and minimum jerk trajectories and solve the resulting two-point boundary value problem.

1 Introduction

There are many applications in which the problem of generating smooth three-dimensional trajectories for a rigid body is encountered. In robotics, it is frequently necessary to plan movements between a given (start) end-effector position and orientation and a desired (goal) position and orientation [1]. In robotics, smooth trajectories are desired because (a) the electro-mechanical system is limited by its actuator size and its control bandwidth so it cannot produce large velocities and accelerations; and (b) movements with high acceleration and/or jerk can excite the structural natural frequencies in the system. The generation of smooth trajectories is also of direct relevance for simulating human motion in computer graphics. Flash and Hogan [2] show that humans plan trajectories that minimize the integral of the Cartesian jerk in point to point reaching motions.

There is extensive literature on trajectory generation in kinematics, robotics and computer graphics [3]. The screw motion is a basis for the schemes in [4], [5] and [6]. Although the screw displacement is invariant with respect to rigid body transformations, it does not optimize a meaningful cost function [7]. Ge and Ravani [8] discuss motion interpolation in an oriented projective space for computer-aided geometric design. A similar approach using quaternions is discussed in [9] for computer animation of rotating rigid bodies. This works do not address the choice of metrics. In contrast, Park and Ravani [10] use the scale-dependent left invariant metric to design Bezier curves for three-dimensional rigid body motion interpolation. Noakes et al. [11] use a bi-invariant metric to define cubic splines on the group of rotations $SO(3)$.

In this paper, trajectories that maximize an appropriate measure of smoothness in the form of an integral cost function are proposed. Depending on the chosen integrand, boundary conditions on the derivatives of the desired order can be enforced. The dynamics of the rigid body can be modeled by choosing the Lagrangian as the cost function. To demonstrate the method we derive necessary conditions for the shortest distance and minimum jerk trajectories. The analytical expressions for the trajectories are computed for some special but important cases.

Because of limitation on space we provide the results without proofs. The interested reader should consult [12] for details and [7] for a discussion on the choice of metrics for $SE(3)$.

2 Kinematics and Lie groups

Consider a rigid body moving in free space. Assume an inertial reference frame F fixed in space and a frame M fixed to the body at point O'. At each instance, the configuration (position and orientation) of the rigid body can be described by a homogeneous transformation matrix corresponding to the displacement from frame F to frame M. The set of all such matrices is called $SE(3)$, the special Euclidean group of rigid body transformations in three-dimensions [13].

On a Lie group, the tangent space at the group identity has the structure of a Lie algebra. The Lie algebra of $SE(3)$ is denoted by $se(3)$ and is given by:

$$se(3) = \left\{ \begin{bmatrix} \Omega & v \\ 0 & 0 \end{bmatrix}, \Omega \in \mathbb{R}^{3\times 3}, v \in \mathbb{R}^3, \Omega^T = -\Omega \right\}.$$

Each 3×3 skew-symmetric matrix Ω can be uniquely identified with a vector $\omega \in \mathbb{R}^3$. An element $T \in se(3)$ can be thus identified with a pair $\{\omega, v\}$. Physically, ω is the angular velocity of the rigid body, while v is the linear velocity of the point O'. The Lie algebra $se(3)$ is thus isomorphic to the set of all twists.

Given a motion of a rigid body described by a curve $A(t)$, a twist T can be computed at each point by:

$$T = A^{-1}\dot{A}. \tag{1}$$

The twist computed in this way consists of velocities that physically correspond to the angular velocity of the rigid body and the linear velocity of the point on the rigid body that is coincident with the origin O' of the frame M, both expressed in the frame M. The twist T is said to have been obtained by the left-translation of the tangent vector $\dot{A}$. It is easy to check that T does not depend on the choice of the inertial frame F.

A set of basis twists which correspond to instantaneous rotations about and instantaneous translations along the Cartesian directions will be used as a basis for $se(3)$ throughout the paper:

$$L_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad L_3 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad L_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad L_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad L_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This basis has the useful property that the components of a twist $T \in se(3)$ are given precisely by the pair of velocities, $\{\omega, v\}$.

The product operation on a Lie algebra is called a Lie bracket. On $se(3)$, the Lie bracket of two elements $t_1 = \{\omega_1, v_1\}$ and $t_2 = \{\omega_2, v_2\}$ is given by

$$\begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \omega_1 \times \omega_2 \\ \omega_1 \times v_2 + v_1 \times \omega_2 \end{bmatrix}. \tag{2}$$

A smooth assignment of a tangent vector to each point of $SE(3)$ is called a vector field. One possible way to define a vector field, X, at an arbitrary element $A \in SE(3)$ is:

$$X(A) = \hat{T}(A) = AT, \tag{3}$$

where $T \in se(3)$. Such vector field is called a left invariant vector field. There is obviously a one-to-one correspondence between the left invariant vector fields and elements of $se(3)$. Since $L_1, L_2, \ldots, L_6$ are a basis for $se(3)$, an obvious choice of the basis for the set of the left invariant vector fields is $\{\hat{L}_1, \hat{L}_2, \ldots, \hat{L}_6\}$, where $\hat{L}_i$ is obtained from L_i according to Equation (3). But at any point the vector fields $\hat{L}_1, \ldots, \hat{L}_6$ are linearly independent, so any vector field X can be expressed as

$$X = \sum_{i=1}^{6} X^i \hat{L}_i, \tag{4}$$

where the coefficients X^i vary over the manifold (if they are constant then X is left invariant). Throughout the paper, we will associate with each vector field X the vector pair $\{\omega, v\}$, defined by

$$\omega = [X^1, X^2, X^3]^T, \qquad v = [X^4, X^5, X^6]^T.$$

3 Riemannian metrics on $SE(3)$

An inner product on $se(3)$ can be extended to a Riemannian metric on $SE(3)$ [14]. Let the inner product of two elements $T_1, T_2 \in se(3)$ be given by

$$< T_1, T_2 >_I = t_1^T W t_2, \tag{5}$$

where t_1 and t_2 are the 6×1 vectors of components of T_1 and T_2 with respect to some basis and W is a positive definite matrix. If V_1 and V_2 are tangent vectors at an arbitrary point $A \in SE(3)$, the inner product $< V_1, V_2 >_A$ in the tangent space $T_A SE(3)$ can be defined by:

$$< V_1, V_2 >_A = < A^{-1}V_1, A^{-1}V_2 >_I . \tag{6}$$

The metric obtained in such a way is called a left invariant metric [14]. A right invariant Riemannian metric can be defined in a similar way.

There is no natural choice of a Riemannian metric on $SE(3)$, but it can be shown that no bi-invariant Riemannian metric exists [15]. In other words, there is no metric that is invariant with respect to changes in inertial frame and with respect to changes in the body fixed frame. Park and Brockett [16] suggested the left invariant Riemannian metric where the matrix W in Equation (5) is given by:

$$W = \begin{bmatrix} \alpha I & 0 \\ 0 & \beta I \end{bmatrix} \tag{7}$$

and α and β are positive scalars which act like scaling factors for angular velocities and linear velocities. In *kinematic analysis* there is no *a priori* justification for choosing them. Because a left invariant metric is independent of the choice of the inertial reference frame, we will use appropriate left invariant metrics in this paper. A more detailed investigation on the choice of possible Riemannian metrics is reported elsewhere [7].

In dynamic analysis, the kinetic energy of a rigid body is a scalar invariant and therefore it makes sense to define the matrix W according to the inertial properties of the rigid body:

$$W = \begin{bmatrix} H & 0 \\ 0 & mI \end{bmatrix}. \tag{8}$$

Matrix H is the inertia tensor of the rigid body and m is its mass. Matrix H can be made diagonal if the body-fixed reference frame is attached at the centroid and its axes are aligned with the principal axes. The metric (8) is also invariant with respect to the choice of the inertial reference frame and the squared norm of a velocity vector at a point equals to the kinetic energy of the rigid body.

Remark: Both metrics, (7) and (8) can be shown to be product metrics (see [12]) so the calculations in the examples could be considerably simplified by performing them on the product space $SO(3) \times \mathbb{R}^3$ rather than on $SE(3)$. However, the key results in this paper are derived for a general metric and are not limited to product metrics. Thus, the derivations do not assume a special product structure for $SE(3)$.

4 Covariant derivative

The motion of a rigid body can be represented by a curve, $A(t)$ on $SE(3)$. The velocity at an arbitrary point is the tangent vector to the curve at that point. In order to obtain other kinematic quantities, such as acceleration and jerk, or to engage in a dynamic analysis, it is necessary to differentiate the velocity vector field along the curve. If the manifold $SE(3)$ is embedded in the space of all 4×4 matrices, differentiation can be carried out in the Euclidean space $\mathbb{R}^{4\times 4}$. Instead, by defining the *covariant derivative* of a vector field, we obtain a derivative that is intrinsic to $SE(3)$ and results that do not depend on the ambient space. Further, such framework is applicable to any Riemannian manifold. The notion of *affine connection* is instrumental in the definition of the covariant derivative. An affine connection basically provides a way to compare tangent vectors that lie in different tangent spaces. If ∇ is the affine connection, the covariant derivative of a vector field V along a curve $A(t)$ is given by $\frac{\mathrm{D}}{dt}V = \nabla_{\frac{dA}{dt}} V$ [14].

If $V(t) = \frac{dA}{dt}$ is the velocity of the rigid body for the motion $A(t)$, the acceleration is given by the covariant derivative of the velocity along the curve

$$\frac{\mathrm{D}}{dt}\frac{dA}{dt} = \nabla_V V$$

The next derivative, jerk, is $\nabla_V \nabla_V V$. Note that the acceleration and jerk (unlike the velocity) depend on the choice of the connection.

4.1 Riemannian connection

Given a Riemannian manifold (a manifold with a Riemannian metric), there exists a unique connection, called Levi-Civita or Riemannian connection, which is compatible with the metric and symmetric. It can be shown [14] that if ∇ is the Riemannian connection, then for any three vector fields X, Y and Z,

$$\begin{aligned}<Z, \nabla_X Y> = \tfrac{1}{2}\{Y<X,Z> + X<Z,Y> - Z<X,Y> + \\ <[Z,Y],X> + <[Z,X],Y> + <[X,Y],Z>\}. \quad (9)\end{aligned}$$

Once we introduce a left invariant metric on $SE(3)$, we can find the corresponding Riemannian connection directly from Equation (9).

Proposition 4.1 *Let $X = X^i \hat{L}_i$ and $Y = Y^i \hat{L}_i$ be two arbitrary vector fields with the corresponding vector pairs $\{\omega_x, v_x\}$ and $\{\omega_y, v_y\}$.*

(a) The Riemannian connection corresponding to the Riemannian metric (7) is given by

$$\nabla_X Y = \{\frac{d\omega_y}{dt} + \frac{1}{2}\omega_x \times \omega_y, \frac{dv_y}{dt} + \omega_x \times v_y\}, \quad (10)$$

where $\frac{d}{dt}$ is the derivative along the integral curve of X.

(b) The Riemannian connection corresponding to the Riemannian metric (8) is given by

$$\begin{aligned}\nabla_X Y = \Big\{ \tfrac{d\omega_y}{dt} + \tfrac{1}{2}\big[(\omega_x \times \omega_y) + H^{-1}(\omega_x \times (H\omega_y)) \\ + H^{-1}(\omega_y \times (H\omega_x))\big], \tfrac{dv_y}{dt} + \omega_x \times v_y \Big\} \quad (11)\end{aligned}$$

It is important to note that Equation (10) is independent of the choice of the constants α and β and that the translational component of $\nabla_X Y$ in (11) is independent of the choice of matrix H and thus the choice of metric on $SO(3)$.

We will also need expressions for the Riemannian curvature [14] corresponding to the Riemannian connection (10).

Proposition 4.2 *If X, Y and Z are three arbitrary vector fields on $SE(3)$ with the associated vector pairs $\{\omega_x, v_x\}$, $\{\omega_y, v_y\}$, and $\{\omega_z, v_z\}$, the Riemannian curvature corresponding to the Riemannian connection (10) is*

$$R(X,Y)Z = \{\frac{1}{4}(\omega_x \times \omega_y) \times \omega_z, 0\} \quad (12)$$

If V is the velocity associated with the motion $A(t)$ of a rigid body and $\{\omega, v\}$ is the corresponding velocity pair, adopting metric (7) the acceleration can be computed from Equation (10):

$$\nabla_V V = \{\dot{\omega}, \dot{v} + \omega \times v\} \quad (13)$$

The third derivative of motion, jerk, can be obtained by considering the covariant derivative of the acceleration along the curve:

$$\nabla_V \nabla_V V = \left\{\frac{d\dot{\omega}}{dt} + \frac{1}{2}\omega \times \dot{\omega}, \frac{d(\dot{v} + \omega \times v)}{dt} + \omega \times (\dot{v} + \omega \times v)\right\}.$$

These expressions can be seen to be the usual expressions for the acceleration and jerk [12].

5 Necessary conditions for optimality

In this section we consider trajectories between a starting and a final position and orientation that minimize integral cost functions while possibly satisfying additional boundary conditions. The cost function can be the kinetic energy of the rigid body, or some other measure of smoothness involving velocity or its higher derivatives. In particular, we will be interested in curves $A : [a, b] \to SE(3)$ that minimize integrals of the form

$$J = \int_a^b < h(\frac{dA}{dt}), h(\frac{dA}{dt}) > dt \qquad (14)$$

where boundary conditions on $A(t)$ and its derivatives may be specified at the end points a and b. The function h returns a vector field and in our case depends on the connection.

The necessary conditions for the optimal trajectories will be derived using calculus of variations on manifolds [11, 14]. We illustrate the basic approach with the simple example in which the cost function is the energy functional:

$$J = L_E = \int_a^b < \frac{dA}{dt}, \frac{dA}{dt} > dt, \qquad (15)$$

and the solution is known to be a minimal geodesic on $SE(3)$ [14, 17].

Let $f(s, t)$ be a variation of $A(t)$, that is $f(0, t) = A(t)$, and let $V = \frac{\partial f(s,t)}{\partial t}$ and $S = \frac{\partial f(s,t)}{\partial s}$. Using the properties of the Riemannian connection, the first variation of L_E can be computed as follows:

$$\begin{aligned} \tfrac{1}{2} L'_E(s) &= \tfrac{1}{2} \tfrac{d}{ds} \textstyle\int_a^b < V, V > dt = \tfrac{1}{2} S \int_a^b < V, V > dt \\ &= \textstyle\int_a^b < \nabla_S V, V > dt \int_a^b < \nabla_V S, V > dt \\ &= \textstyle\int_a^b (V < S, V > - < S, \nabla_V V >) dt \\ &= < S, V > |_a^b - \textstyle\int_a^b < S, \nabla_V V > dt. \end{aligned} \qquad (16)$$

Equation (16) must be satisfied for an arbitrary variational vector field S, so if $A(t) = f(0, t)$ is a critical point:

$$\nabla_V V = 0, \qquad (17)$$

where $V = \frac{dA(t)}{dt}$. We thus obtain, as expected, the equation defining a geodesic.

To solve Equation (17) and find the geodesics on $SE(3)$, we express V as a linear combination of left invariant vector fields $\hat{L}_1, \ldots, \hat{L}_6$ according to Equation (4).

Proposition 5.1 *If $A(t)$ is a geodesic for the metric (7), the vector pair $\{\omega, v\}$ corresponding to the velocity vector field $V = \frac{dA}{dt}$ must satisfy the equations:*

$$\frac{d\omega}{dt} = 0 \qquad \frac{dv}{dt} = -\omega \times v. \qquad (18)$$

It is worth noting that the above result is independent of the choice of the scale factors α and β. The necessary conditions for the minimum jerk curves derived in the next subsection will also have the same property.

5.1 Minimum jerk curves

The minimum jerk curve between two points is obtained by minimizing the integral of the norm of the Cartesian jerk, provided that the appropriate boundary conditions are given. In particular, minimum jerk trajectories are well defined when the initial and final velocities and accelerations are specified. Such trajectories are particularly useful in robotics where one is generally able to control the acceleration of the end effector of a robot (and therefore the velocity and position) but the electro-mechanical actuators cannot produce sudden changes in the acceleration. It is interesting to note that Flash and Hogan [2] suggest that humans plan trajectories that minimize such an integral measure of the jerk when reaching from one point to another.

The minimum-jerk cost functional is:

$$J = L_J = \int_a^b < \nabla_V \nabla_V V, \nabla_V \nabla_V V > dt \qquad (19)$$

where as before $V = \frac{dA(t)}{dt}$. The curve must start and end at the desired points on the manifold and with the desired velocities and accelerations. We arrive at the necessary conditions for the solution by following the same approach as in the previous subsection.

Theorem 5.2 *Let $A(t)$ be a curve on a Riemannian manifold with the metric (7) that satisfies the boundary conditions (that is, it starts and ends at the prescribed points with the prescribed velocities and accelerations) and let $V = \frac{dA}{dt}$. If $A(t)$ minimizes the functional L_J (Equation 19), then:*

$$\nabla_V^5 V + R(V, \nabla_V^3 V)V - R(\nabla_V V, \nabla_V^2 V)V = 0. \qquad (20)$$

5.2 Minimum energy curves

In this subsection, we consider the kinetic energy of the rigid body as a cost function and determine trajectories that minimize the energy over the entire trajectory. We therefore compute the geodesics corresponding to the metric (8).

Proposition 5.3 *If $A(t)$ is a geodesic for the metric (8), the vector pair $\{\omega, v\}$ corresponding to the velocity vector field $V = \frac{dA}{dt}$ must satisfy the equations:*

$$\frac{d\omega}{dt} = -H^{-1}(\omega \times (H\omega)) \qquad \frac{dv}{dt} = -\omega \times v. \qquad (21)$$

6 Solutions for optimal trajectories

6.1 Shortest distance path on $SE(3)$

Using properties of Riemannian covering maps, Park showed [17] that the geodesics on $SE(3)$ can be obtained by lifting the geodesics from $SO(3)$ and $\mathbb{R}^3$. We derive this result directly from Equation (18).

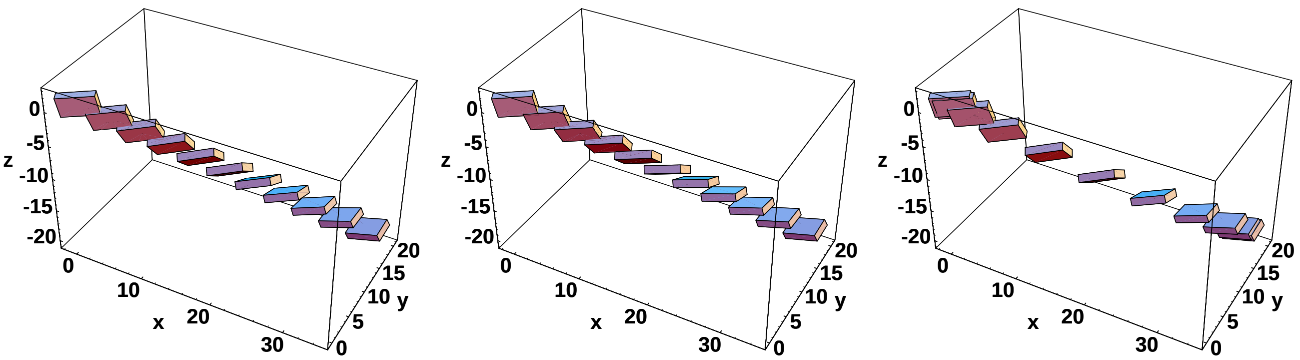

Figure 1: Trajectory of an object following: (a) the shortest distance path (no boundary conditions on the velocities and accelerations); (b) the minimum energy path; and (c) the minimum jerk motion (with zero velocities and accelerations at both end points).

Proposition 6.1 *Given two positions and orientations*

$$A_1 = \begin{bmatrix} R_1 & d_1 \\ 0 & 1 \end{bmatrix} \quad \textit{and} \quad A_2 = \begin{bmatrix} R_2 & d_2 \\ 0 & 1 \end{bmatrix}$$

the shortest distance path (geodesic) is given by

$$R(t) = R_1 \exp(\Omega_0 t) \qquad d(t) = t(d_2 - d_1) + d_1 \qquad (22)$$

where

$$\Omega_0 = \log(R_1^T R_2)$$

and exp *and* log *are matrix exponential and logarithm (see [13]).*

It is interesting to compare this shortest distance trajectory to the trajectory that minimizes the kinetic energy. According to the Hamilton's principle [18] for holonomic systems a trajectory that minimizes the integral of the energy is given by the dynamic equations of motion. We found the necessary conditions for this trajectory in Equation (21). The rotational part in Equation (21) are the Euler equations which, in general, do not admit an analytical solution. However, it is easy to compute the translational trajectory of the rigid body: The point O' (which was chosen to coincide with the center of mass) travels in a straight line with a uniform velocity between the initial and final position of the rigid body. This is true for an arbitrary choice of the positive-definite matrix H and as a special case contains the solution for the metric (7) obtained in Equations (22). The result is also the geometric statement of Newton's second law: The center of mass of the rigid body (the origin O') travels in a straight line if no external force acts on the body.

The shortest distance trajectory given by Equations (22) and the dynamically-correct trajectory given by Equation (21) for a homogeneous rectangular prism ($5 \times 1 \times 3$ cubic units) are shown in figures 1(a) and 1(b). Figure 1(a) shows the motion along the shortest distance path (the geodesic) given by Equations (22). The center of mass of the object moves along the straight line connecting its initial and final position while the body rotates from its initial to its final orientation. Figure 1(b) shows the trajectory that the rigid body would follow according to the principles of rigid body dynamics if it were launched with the appropriate initial condition and were not subject to external forces. This is a geodesic for the metric defined in Equation (8). The center of mass again travels along the straight line, but the body rotation is governed by the Euler (dynamic) equations. The trajectory was computed numerically by solving a two-point boundary value problem. See [19] for a discussion of the numerical methods. Note that the shape and mass distribution of the rigid body is only relevant to the trajectory in Figure 1(b). Finally, it is worth noting that these trajectories are very different from the screw motion between the initial and final position [12].

6.2 Minimum jerk trajectories with the homogeneous boundary conditions

In general, the rotational components of the necessary conditions for the minimum jerk curves (Equation 20) cannot be solved analytically. However, in the special case when the initial and final velocities and the initial and final accelerations are prescribed to be 0, an analytical solution of these equations can be obtained.

Proposition 6.2 *The minimum jerk trajectories in the case when the initial and final velocities and the initial and final accelerations are prescribed to be* 0 *are given by*

$$R(t) = R_1 \exp(p(t)\Omega_0) \qquad d(t) = p(t)(d_2 - d_1) + d_2,$$

where $p(t) = 6t^5 - 15t^4 + 10t^3$ *and the constants* Ω_0, d_1 *and* d_2 *can be determined from the initial and final positions.*

Corollary 6.3 *When there are homogeneous boundary conditions in velocities and accelerations, the minimum jerk trajectories follow the same path as the shortest distance path, only the parameterization changes.*

The minimum jerk trajectory for the previous example is shown in Figure 1(c). Comparing this trajectory with the minimum distance trajectory in Figure 1(a), it is evident that the path followed by the rigid body is the same. However, the minimum jerk trajectory must start and end with zero velocity and zero acceleration. Therefore, the velocity starts from zero with zero acceleration, rises to a peak and then decreases to zero with zero acceleration.

7 Conclusion

This paper addresses the general problem of generating kinematically and dynamically optimal trajectories for a rigid body between an initial and a final position and orientation while satisfying boundary conditions on the derivatives at these positions. The problem is formulated as a variational problem on the Lie group of rigid body motions $SE(3)$. We define a metric on the Lie algebra $se(3)$ leading to a left invariant Riemannian metric, which in turn gives rise to a Riemannian connection on $SE(3)$. We show that the acceleration computed using this connection corresponds to the acceleration of the rigid body. We formulate the geodesic equation and analytically compute its solution by expressing the tangent vector field in the appropriate basis. Finally, we show how the necessary conditions for the cost functions that involve the norm of any vector-valued function can be computed. We present solutions for the extremals when the cost functions are distance, energy and the norm of the jerk.

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