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Stable Haptic Interaction with Switched Virtual Environments

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Abstract

This paper investigates haptic interaction with virtual environments composed of objects with diverse dynamic properties. Passivity is often used for stability analysis of haptic systems. We demonstrate that when the dynamics of the virtual environment during the interaction changes, the approaches using traditional notion of passivity fail. We then show that a recently proposed notion of passivity for hybrid systems can be used to design stable interaction strategies for such systems.

1 Introduction

Stability of haptic systems and a closely related problem of bilateral manipulation has been of considerable interest to many researchers. One of the major obstacles to stability in such systems proved to be time delay, either in the form of communication delay or due to computation. Passivity emerged as an especially powerful paradigm to study stability of both linear and non-linear systems with time-delay [1, 2, 3, 4]. It has been shown that several other factors also affect the stability in haptic systems [5], but passivity is still well suited for their analysis. In [6] and [7], passivity has been used to analyze the stability of haptic displays interacting with linear and passive environments. Passivity analysis based on linear circuit theory was used for stability analysis in [8]. The most general stability results for haptic systems are derived in [9] where passivity is shown to be appropriate for the analysis of both passive and non-passive, and linear and non-linear environments than can be implemented using either implicit or explicit numerical methods. In this analysis, the lack of passivity in the virtual environment is compensated by designing so called virtual coupling such that makes the haptic system passive and hence stable. A new method that actively controls the energy flow into the system in order to guarantee passivity is described in [10].

In majority of works above, a single non-linear function is proposed to describe the dynamics of a virtual environment (see e.g. [9]). However, a complex virtual environment might not be suitably represented in this way. Representation by multiple functions is often more natural and better reflects the nature of the interaction of the user with the virtual environment. Consequently, such haptic systems are hybrid in the sense that when the user touches different objects in the virtual environment, she is switching between different dynamic behaviors. A key issue involved in such switching is that of stability.

So far the hybrid nature of haptic systems has not been acknowledged. This can be attributed to the lack of techniques for the design of hybrid controllers, a subject of considerable research [11, 12, 13]. On the other hand, stability of hybrid systems has been well studied and several results exists [14, 15, 16, 17, 18, 19]. The aim of this paper is to show how these results can be applied to haptic systems. In particular, we show that a notion of passivity for hybrid systems developed in [20] can be used to design stable haptic interaction strategies for complex virtual environments.

The paper is organized as follows. We first briefly review the definition of passivity for continuous systems and describe how it applies to haptics. We show why such an approach fails if the user haptically interacts with objects that have different dynamic properties. We then discuss the notion of passivity for hybrid systems and show how it can be used to design stable haptic interaction strategies. We conclude the paper with an example.

2 Passivity and its application to haptics

A system defined by:

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned} \tag{1}$$

where $f(0,0) = 0$ and $h(0,0) = 0$ is *passive* if there exists a $\mathcal{C}^1$ positive semidefinite function $V(x)$ (called the storage function) such that:

$$u^T y \geq \frac{dV}{dt} + \varepsilon u^T u + \delta y^T y \quad \forall (x,u) \tag{2}$$

where ε and δ are nonnegative constants. One can define the passivity of sampled-data systems similarly: a sampled-data system is passive if there exists a discrete storage function W such that

$$u(k)^T y(k) \geq \Delta W(k) + \varepsilon u(k)^T u(k) + \delta y(k)^T y(k)$$

where $\Delta W(k) = W(k+1) - W(k)$, and ε and δ are nonnegative constants. In both cases, if $\varepsilon > 0$ the system is *input*

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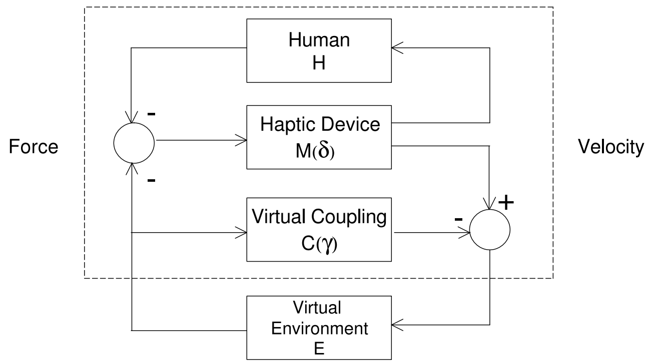

Figure 1: Graphical representation of the haptic display.

strictly passive (ε-ISP), and if $\delta > 0$ the system is *output strictly passive* (δ-OSP).

The intuitive interpretation of this definition is that passive systems can not generate energy on their own. It can be shown that if the system is passive and the storage function is positive definite, the origin is a stable equilibrium point. In this case the storage function V becomes a Lyapunov function. What makes passivity further useful for the stability analysis is that, loosely speaking, an interconnection of passive systems is again passive. This observation has been the basis for the stability proofs in [1, 2, 6, 7, 8, 9].

A concept that proved useful in the design of haptic systems is so called virtual coupling [21]. Virtual coupling decouples the haptic device control problem from the virtual scene generation. A schematic of a haptic display connected to a virtual environment through the virtual coupling is shown in Figure 1 [9]. In the figure, H is the human operator, M is the haptic device, C is the virtual coupling, and E is the virtual environment. A common assumption in the stability analysis of haptic systems is that the human behaves passively. Assume that the human and the haptic device are δ-OSP and the virtual coupling is γ-OSP. Let α be the amount by which the virtual environment lacks ISP. It was shown in [9] that if the virtual coupling is designed so that

$$\gamma = \frac{\delta\alpha}{\delta - \alpha}, \tag{3}$$

then this class of non-linear non-passive environments can be displayed stably for all passive H.

It is natural to assume that the virtual coupling and the virtual environment are discrete-time systems, while the human-device subsystem is a continuous system. If the human-device subsystem is controlled with a discrete-time controller, the passivity (and therefore stability) of the overall system can be analyzed in the discrete-time domain. Again, it was shown in [9] that the system will be passive if in addition to Eq. (3) it holds that

$$\delta > \frac{\sigma T}{2},$$

where σ is the magnitude of the largest negative slope of the non-linear function describing the stiffness of the virtual environment and T is the sampling time of the human-device system.

3 Complex virtual environments

Representing the dynamics of the virtual environment by several functions is often more natural and better reflects the nature of the virtual environment than a single non-linear block. For example, an environment may consist of objects whose dynamics are unrelated. Furthermore, a virtual coupling designed for a complex virtual environment will be quite conservative since it needs to be designed for the least passive object in the virtual environment. Since the virtual environment introduces distortion in the haptic perception of the environment it would seem natural to have a different virtual coupling for each object and then switch between them when the user interacts with different objects. For example, suppose we are simulating a device such as a phone having different buttons and we want the feel of one button to be different than those of the other buttons. Since such a system needs to switch between different dynamic regimes in response to the user action it can be modeled as a **hybrid system**.

The following example shows that while having different virtual couplings promises better performance, care is needed when implementing such hybrid virtual environments. Consider a virtual environment made up of two regions with different dynamic properties. When the human interacts with a particular region, the virtual environment that describes that region is switched on. The state of the virtual environment that switched *off* is passed to the system that is switched *on*. The dynamics describing each region is second order and is constructed so that each region lacks ISP by a different amount. The user interacts with the virtual environment through a 1-DOF haptic display as depicted in Figure 1. A virtual coupling is designed for each environment to guarantee that the interaction with that environment is stable [9]. The virtual coupling is assumed to be just a pure damper but more complex dynamics can be chosen at the cost of increasing the order of the overall system. The human is assumed to be passive and is modeled as a pure damper having a damping b'. The device has a mass m and a damping b. Since the human is connected to the device in a negative feedback loop, the $H+D$ subsystem depicted in Figure 2 can be modeled as an equivalent system having a mass m and a damping $b+b'$.

The system parameters for the example were chosen to be:

- Mass of the haptic device: $m = 0.01$ kg
- Combined damping of the human-device: $b + b' = 12\ \mathrm{Nm^{-1}s}$

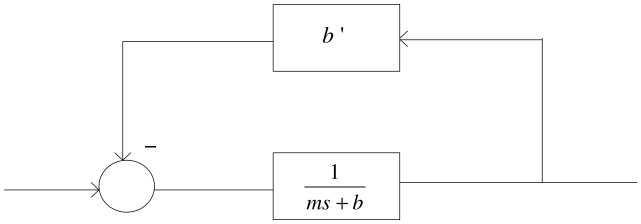

Figure 2: Negative feedback supplied by human to the haptic device

- Damping of the virtual coupling: $b^* = 12$ Nm^{-1}s
- Sampling time: $T = 0.001$ s

Note that in this example **the same virtual coupling is used for both environments**. In other words, Eq. (3) is satisfied with the same γ for both environments thereby ensuring the stability of the interaction with each environment.

By assuming the mass m to be small, we are effectively having the $H+D$ sub-system (human-device) resemble a massless damper. If x_1 is the velocity of the haptic device, the equations describing the system (see Figure 2) are:

$$\begin{aligned} \dot{x}_1 &= -\frac{(b+b')}{m}x_1 + \frac{u}{m} \\ y &= x_1 + \frac{u}{b^*} \end{aligned} \quad (4)$$

where u is the force provided by a virtual environment and y is the combined velocity of the haptic device and the virtual coupling.

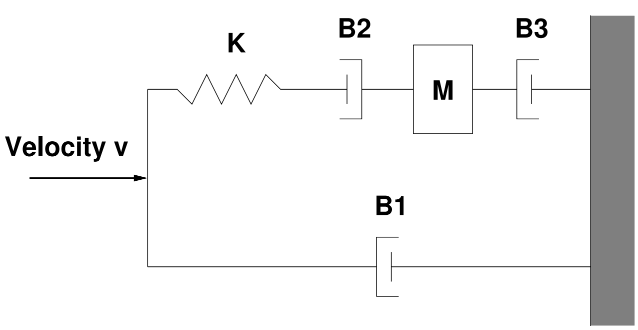

Figure 3: Mechanical equivalent of the two virtual environments.

The model for the two virtual environments is shown in Figure 3. The complete state of the environment can be described by the position and velocity of the mass M. However, for convenience the state variables x_2 and x_3 used to describe the system were chosen to be the force generated by the spring K and the velocity of the mass M, respectively. With these state variables, the equations describing the system are:

$$\begin{aligned} \dot{x}_2 &= -\frac{K}{B_2}x_2 - Kx_3 + Ku \\ \dot{x}_3 &= \frac{x_2}{M} - \frac{B_3}{M}x_3 \\ y &= x_2 + B_1 u \end{aligned} \quad (5)$$

The output y of this system is the force which is in turn the input to the top sub-system. The input u is the combined velocity of the haptic device and the virtual coupling.

3.1 Virtual Environment A

The parameters chosen for the virtual environment A were $M = 0.02$, $K = 20$, $B_1 = -0.11$, $B_2 = 20$, and $B_3 = 0.01$. The damper B_1 is negative so the virtual environment lacks passivity. In applications, a negative resistance could be the result of a virtual guide, for example. Substituting these values into Eq. (5) results in the following system:

$$\begin{aligned} \begin{bmatrix} \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} -1 & -20 \\ 50 & -0.5 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 20 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} - 0.11u \end{aligned} \quad (6)$$

Using the Kalman-Yakubovich-Popov lemma [22], it can be easily checked that for this virtual environment the overall haptic system is passive and therefore stable. A trajectory of the system in the x_2x_3 plane for the initial condition $\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$ is shown in Figure 4.a.

3.2 Virtual Environment B

The parameters chosen for the virtual environment B were $M = 0.05$, $K = 50$, $B_1 = -0.11$, $B_2 = 100$, and $B_3 = 0.05$. As before, the damper B_1 is negative and the system lacks passivity. The system equations in this case are:

$$\begin{aligned} \begin{bmatrix} \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} -0.5 & -50 \\ 20 & -1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 50 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} - 0.11u \end{aligned} \quad (7)$$

A quick analysis shows that also for this virtual environment the overall haptic system is passive and thus stable. A trajectory of the system in the x_2x_3 plane for the initial condition $\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$ is shown in Figure 4.b.

3.3 Switching rules

We also need to define the rules that govern when a particular virtual environment is active. Consider a switching sequence defined by:

The virtual environment A is active when $x_2x_3 \geq 0$ *and the virtual environment B is active when* $x_2x_3 < 0$.

The result of such a switching is that the virtual environment A is active in the quadrants I and III of the x_2x_3 plane while the virtual environment B is active in the quadrants

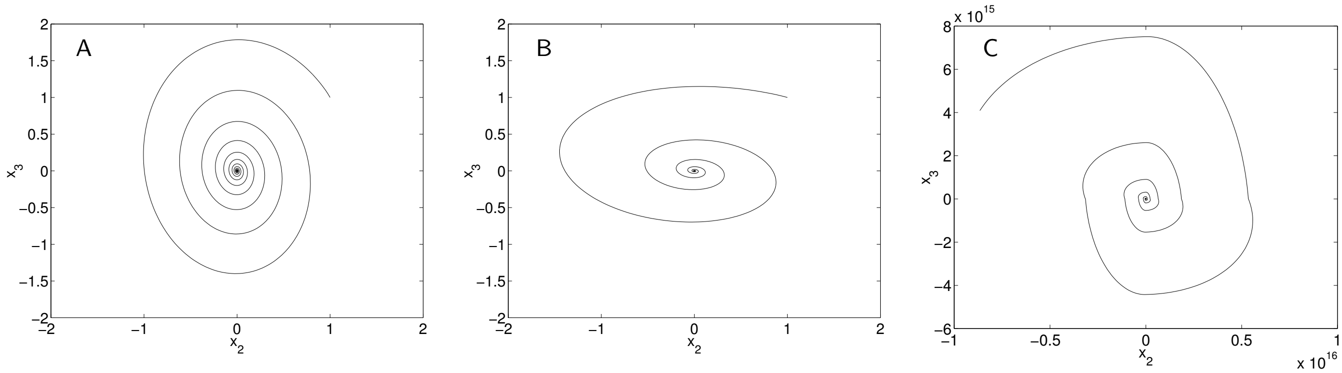

Figure 4: (a) Virtual Environment A is stable. (b) Virtual Environment B is stable. (c) Switching between virtual environments A and B drives the system unstable.

II and IV of the x_2x_3 plane. A trajectory of the system for these switching rules for the initial condition $\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$ is shown in Figure 4.c. It is clear that the system is unstable.

Therefore, even if the concept of passivity and the energy considerations that lead to stability are intuitive and therefore appealing, the concept might be misleading when dealing with hybrid systems. One might be lead to believe that if the system switches between two sets of state equations that are both passive, the resulting hybrid system must also be passive. The example clearly demonstrates that such a conclusion would be wrong. In order to conclude that the resulting system is passive, **the storage function in** (2) **would have to be the same for both systems**. This is not true in the example and in general is difficult to guarantee when designing systems that can switch between different (passive) dynamic behaviors. The next section describes a framework that can be used to study such systems.

4 Passivity of Hybrid Systems

Formally, a hybrid system is a tuple:

$$\mathrm{HS} = (\Xi, \mathcal{M}, \Gamma, \mathcal{U}, \Sigma, \mathcal{F}, \mathcal{H}) \tag{8}$$

where $\Xi \subset \mathbb{Z}$ is a (finite) set of discrete states, $\mathcal{M} = \{M_i\}_{i\in\Xi}$ is a collection of (differentiable, connected) manifolds, $\Gamma \subset \mathbb{Z}$ is a set of discrete inputs, $\mathcal{U} \subset \mathbb{R}^m$ is a set of continuous inputs, $\mathcal{F} = \{f_i\}_{i\in\Xi}$ is a set of ($\mathcal{C}^1$) control vector fields $f_i : M_i \times \mathcal{U} \to TM_i$, $\Sigma : \Xi \times \mathbb{R}^n \times \Gamma \times \mathcal{U} \to \Xi$ is a function describing the discrete evolution of the system, and $\mathcal{H} = \{h_i\}_{i\in\Xi}$ is a set of ($\mathcal{C}^1$) output maps $h_i : M_i \times \mathcal{U} \to \mathbb{R}^m$. The system evolves on M_i following the vector field f_i as long as $\Sigma(i, x, \eta, u) = i$. When $\Sigma(i, x, \eta, u)$ becomes equal to $j \neq i$, the system dynamics switches to (M_j, f_j). In this paper we assume that the state does not change during the switch. Also, we assume that there are finitely many switches in any finite time interval (we exclude phenomena like chattering). The value of $\Sigma(i, x, \eta, u)$ can change either autonomously , e.g. if the trajectory of the system leaves the manifold M_i and enters M_j, or due to a control, e.g. because the discrete input η changes.

It has been shown in [14, 23, 24, 15] that it is not necessary to find a global Lyapunov function in order to guarantee that a hybrid system is stable; it suffices to analyze the stability in each dynamic regime (M_j, f_j) and the switching behavior of the system. Furthermore, it is known that the storage function of a passive system is a candidate Lyapunov function for stability analysis. This suggests that the passivity for hybrid systems should be defined in terms of storage functions of the individual discrete regimes; requiring that a single global storage function exists is too restrictive. A notion of passivity for hybrid systems based on this observation has been suggested in [20].

Definition 1 *Take a hybrid system* (8) *such that for every regime i, $0 \in M_i$ and $f_i(0,0) = 0$. Such a system will be called a **passive hybrid system (PHS)** if the following two conditions hold:*

1. *Each discrete regime (M_i, f_i) is passive. That is, there exists a storage function V_i and $\varepsilon_i, \delta_i \geq 0$ such that*

$$u^T y \geq \frac{dV_i}{dt} + \varepsilon_i u^T u + \delta_i y^T y \quad \forall (x,u) \tag{9}$$

   where (x,u) is a trajectory of (M_i, f_i).

2. *The storage functions have the property that:*

$$V_i(x(t_{i,k-1})) + \int_{t_{i,k-1}}^{t_{i,k}} u^T y dt \geq V_i(x(t_{i,k})). \tag{10}$$

   where $t_{i,k}$ denotes the k-th time that the vector field f_i becomes "active", i.e., $\xi(t_{i,k}^-) \neq \xi(t_{i,k}^+) = i$.

Note that Eq. (9) has to hold whenever the system is in the regime i, whereas the integral in Eq. (10) runs over the regimes that the system traverses before switching back to i. In [20], two important results have been derived: (a) if all the storage functions of a PHS system are positive definite, then the system is stable for $u(t) = 0$; and (b) a feedback interconnection of two PHS's is again a PHS. The notion of PHS therefore plays the same role for hybrid systems as does the

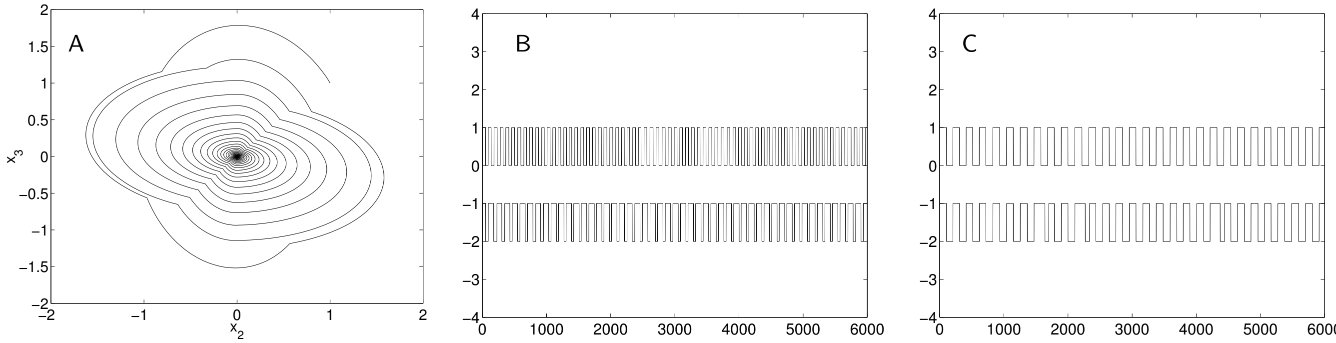

Figure 5: (a) The system is stable when the condition in Eq. (10) is enforced during switching. (b) Switches are missed if they are attempted every 0.04s. (c) No switches are missed if they are attempted every 0.1s. On (b) and (c), the upper curve is the desired switching sequence, the lower the actual switching sequence.

classical passivity for continuous systems. Clearly, an analogous definition of PHS can be formulated for sampled-data systems.

5 Stable implementation of switched virtual environments

In this section we describe an implementation of the system described in Section 3 that guarantees the stability of the system. As mentioned above, one possible solution would be to design a single virtual coupling that would make the system passive for both virtual environments and would guarantee the existence of a **common positive definite storage function**. The general conditions for the existence of a common Lyapunov function (which is closely related to the storage function) for switched linear systems are given in [25]. On the other hand, it can be shown that **the design of a controller (virtual environment)** that simultaneously stabilizes several linear systems results in a bilinear matrix inequality and is thus computationally hard. In any case, it is not difficult to see that none of these methods can produce a desired virtual environment for our example.

An alternative is to enforce the system to be PHS. This will automatically guarantee that the system is stable. Therefore, we need to make sure that Eqs. (9) and (10) are always satisfied. The first equation simply requires that a virtual coupling satisfying Eq. (3) be designed for each virtual environment. We already did this in Section 3. On the other hand, Eq. (10) imposes restrictions on switches between virtual environments: the desired switch can not take place until that condition is satisfied. In other words, even if the action of the user triggers a switch, the system needs to wait until Eq. (10) becomes satisfied for the switch to actually take place.

The simulation results for the system in Section 3 modified to implement the proposed strategy are shown in Figure 5.a. Clearly, the system is stable. It is also apparent that the delay in the switching is about $\frac{1}{8}$ of the time constant for the (x_2, x_3) subsystem. Characterizing this delay for different virtual environments is important for evaluating the performance of the proposed strategy and will be undertaken in future work.

An important issue that needs to be addressed when implementing the proposed stabilization strategy is how to deal with switches that are close in time. Since each individual virtual environment is passive and can be made to asymptotically converge to the origin it can be shown that if a switch has been triggered, the condition Eq. (10) will eventually be satisfied and the switch will take place. However, the user might trigger another switch during the time the system is waiting for the previous switch to take place. The system designer needs to make a choice between two possible policies: (a) the system either waits until the previous switch has taken place before trying to implement the newly arrived switch request; or (b) the switch that has not yet taken place is skipped and the system tries to implement newly arrived switch request. Each policy has it merits and drawbacks. The policy (a) implements the ideal case better since the sequence of environments through which the system evolves equals the desired sequence. However, under this policy the delays might start to accumulate and the system might eventually become completely desynchronized with the user actions. The policy (b) keeps the system always synchronized with the user actions, however some switches never take place and as a result the executed trajectory might miss important actions that the user intended to perform (like pressing a button, for example).

Figures 5.b and 5.c demonstrate the performance of the system if the strategy (b) is used. In the simulation shown in Figure 5.b, the user attempted to switch between the two virtual environments every 40 samples (every 0.04s). The figure shows that some switches are missed. Figure 5.c shows the case when the user attempts to switch between the virtual environments every 100 samples (0.1s). In this case, no switches are missed.

6 Conclusion

In this paper, we investigated stability of haptic interaction with complex virtual environments. The representation of such a virtual environment by a single dynamic regime may be inadequate or impractical for many applications. A representation in which different dynamic behaviors are used to describe different responses of the virtual environment are often more appropriate. However, we demonstrated on an example that even if a virtual coupling is designed for each type of the virtual environment so that the system is passive (and thus stable), switches between different virtual environments might drive the system unstable. We proposed a strategy based on the notion of passivity for hybrid systems that guarantees stability of the interaction with such switched virtual environments. The main characteristic of the proposed strategy is that a switch can not occur at the time the user triggers it, a delay is introduced in order to preserve the stability of the system. A brief discussion on the performance of the scheme is presented, but more thorough investigation shall be considered in future work.

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