A variational calculus framework for motion planning

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Abstract

In this paper, we argue that it is advantageous to formulate motion planning as a variational problem. In this way, a unified framework for addressing redundancy, constraints, and optimality is obtained. An efficient numerical method is proposed for solving the motion planning problem in the variational form. The approach is illustrated on the problem of motion planning for systems that change dynamic equations at discrete time instants as they move. A technique is described for computing a motion plan when the sequence of dynamic equations describing the motion is known. Finally, a motion plan for a simple grasping task, a typical representative of this class of systems, is computed.

1 Introduction

In most of the literature, motion planning is defined as finding a suitable path between two points in space [1]. We propose a broader view in which a motion plan consists of every aspect of motion that is needed to perform the task. Figure 1 shows components of a motion plan as the system and the task become more complex.

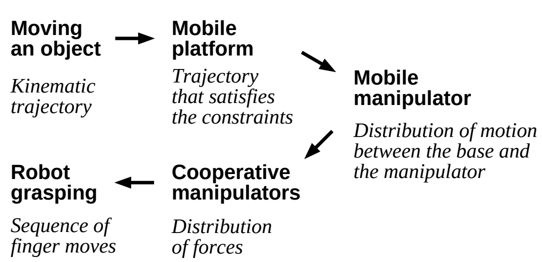

Figure 1: Components of a motion plan.

We argue that the motion planning problem is typically underdetermined and to resolve this indeterminacy it makes sense to define a measure of performance and find a plan with the best performance. In turn, we formulate motion planning as a variational problem thereby obtaining a unified framework that has several advantages over other motion planning methods: (a) We can account for the dynamics of the system; most other methods only provide kinematic trajectories. (b) Kinematic and actuator redundancies can be resolved simultaneously; other approaches employ different strategies at different levels. (c) We can deal with equality and inequality constraints, including nonholonomic constraints. (d) The method yields a feasible and globally optimal solution; kinematic motion planning methods and local methods for redundancy resolution might produce solutions that are not suitable for implementation.

Optimal control and variational calculus have been extensively used for motion planning in the robotics literature (e.g., [2]-[7]). Most of these works concentrate on a particular aspect of motion, such as time-optimality or resolution of redundancy, they do not use variational calculus as a general framework for motion planning.

The variational formulation of the motion planning problem studied in this paper is:

Problem 1.1 (Motion planning) *Given the dynamic equations of the robot:*

$$\dot{x} = f(x, u, t), \tag{1}$$

(x is the vector of state variables and u is the vector of inputs), equality and inequality constraints:

$$g_i(x, u, t) = 0 \qquad i = 1, \ldots, k \tag{2}$$

and

$$h_i(x, u, t) \leq 0 \qquad i = 1, \ldots, l, \tag{3}$$

the desired initial and final configurations:

$$\alpha(x, t)|_{t_0} = 0 \quad \text{and} \quad \beta(x, t)|_{t_1} = 0, \tag{4}$$

and given a cost functional in the form:

$$J = \Psi(x(t_1), t_1) + \int_{t_0}^{t_1} L\left(x(t), u(t), t\right) dt, \tag{5}$$

find (a piecewise smooth) state vector $x^*(t)$ and (a piecewise continuous, bounded) input vector $u^*(t)$ that satisfy equations (1)-(4) and minimize the cost functional (5).

A novel numerical method for solving such variational problems, which combines a discretization of the continuous problem motivated by the finite-element methods with the techniques from nonlinear programming, is described in the first part of the paper. In the second part of the paper, we show that the method can be used to find a motion plan for tasks like grasping and walking, where the dynamic equations change at discrete points in time as the system moves. The state space for such tasks gets partitioned in different regions, each of them characterized by a different set of dynamic equations. While it is difficult to obtain the optimal sequence of regions that the system should traverse, we present a technique for computing the motion plan once this sequence is selected.

2 Numerical method

Consider the Bolza form [8] of the variational problem[1]:

$$\min_x \int_{t_0}^{t_1} L(x, \dot{x})\ dt.$$

subject to

$$\varphi(x, \dot{x}) = 0, \qquad g(x) = 0. \tag{6}$$

For now we assume that only equality constraints are present. Inequality constraints can be treated similarly using slack variables [9].

To find a numerical solution, we approximate the unknown (vector) function with a set of basis functions:

$$x_i(t) \approx \sum_{j=0}^{N} p_i^j \phi_j(t). \tag{7}$$

The approximation of the function is required to be exact on the chosen set of grid points:

$$x_i(a_j) = \sum_{j=0}^{N} p_i^j \phi_j(a_j). \tag{8}$$

For simplicity, we assume that $a_j - a_{j-1} = h$ for all $j = 1, \ldots, N$.

There are many possible choices for the functions ϕ_j, but the computation of the gradient needed for minimization is simplified if the shape functions have localized support[2]. For our computations, we choose the triangular shape functions:

$$\phi_k(t) = \begin{cases} \frac{t-a_{k-1}}{h} & \text{if} \quad a_{k-1} < t \leq a_k, \\ \frac{a_{k+1}-t}{h} & \text{if} \quad a_k < t \leq a_{k+1}, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

In this way, the function x is approximated by a piecewise linear function.

The last equation implies that in Eq. (7), $p_i^j = x_i^j$, where x_i^j is the value of the unknown x_i at the grid point a_j. Let $\hat{x}$ denote the approximation of the function x, and let $\bar{x}$ be the collection of the values of x at the grid points. Discretization in (7) leads to the following finite-dimensional nonlinear programming problem:

$$\min_{\bar{x}} \int_{t_0}^{t_1} L\left(\hat{x}, \dot{\hat{x}}\right)\ dt \tag{10}$$

subject to

$$\int_{a_{j-1}}^{a_j} \varphi(\hat{x}, \dot{\hat{x}})\, dt = 0 \qquad g(\hat{x}(a_j)) = 0 \qquad j = 1, \ldots, N.$$

The resulting constrained nonlinear programming problem can be solved using any of the methods available in the literature. In this work we use the method of augmented Lagrangian [9] to convert the constrained problem into an unconstrained one and Newton's minimization method to solve the ensuing unconstrained minimization problem. We refer the reader to [10] for details.

3 Motion planning for systems with discrete and continuous state

Most motion planning methods assume that the dynamic equations of the system do not change during the task. In many robotic applications this assumption does not hold. For example, for a multi-fingered hand holding an object, any time a new finger is placed on the object or one of the fingers currently in contact with the object is withdrawn, the dynamic equations and the algebraic constraints describing the state of the system change. The state space of such systems can be partitioned into regions so that in each of the regions the system is described with a different set of equations. These regions can be viewed as discrete states; within each discrete state, the differential equations describe the evolution of the continuous state. Systems with discrete and continuous state form a subset of the class of so called hybrid systems [11, 12].

3.1 Mathematical formulation

Let the continuous state space $\mathcal{X}$ of the dynamical system with n states be given by:

$$\mathcal{X} = \bigcup_{j=1}^{p} D_j, \tag{11}$$

where D_j are pairwise disjoint, connected subsets of $\mathbb{R}^n$. On each subset D_j, the system is described with system equations:

$$\dot{x} = F_j(x, u), \tag{12}$$

and algebraic constraints:

$$G_j(x, u) = 0, \qquad H_j(x, u) \leq 0. \tag{13}$$

The vector $x \in \mathcal{X} \subset \mathbb{R}^n$ is the (continuous) state of the system, $u \in \mathbb{R}^m$ is the input, F_j is a (smooth) vector field and G_j and H_j are smooth (vector) functions. The sets D_j are called discrete states. We require that the continuous state changes continuously (but not smoothly) between the regions.

Since the sets D_j are disjoint, given a continuous

[1] If the time t explicitly occurs it can be taken to be an additional state variable so that the system becomes autonomous.

[2] Support of a function is the set $\mathrm{Supp} f = \mathrm{Cl}\{x | f(x) \neq 0\}$, where Cl stands for closure.

state x, there is a unique set $D_{j(x)}$ such that $x \in D_{j(x)}$. Equations (12)-(13) can be therefore rewritten as:

$$\dot{x} = f(x, u) \tag{14}$$

and

$$g(x, u) = 0, \qquad h(x, u) \leq 0, \tag{15}$$

where $f(x, u) = F_{j(x)}(x, u)$, $g(x, u) = G_{j(x)}(x, u)$, and $h(x, u) = H_{j(x)}(x, u)$.

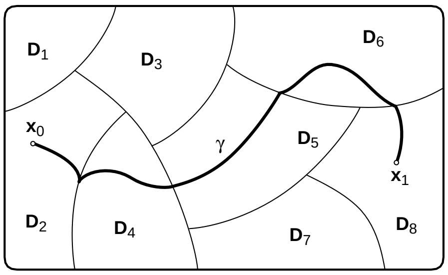

Figure 2: A trajectory of a system with discrete and continuous state.

Consider the motion planning problem for the system (14). Our premise is that the task provides a way to evaluate the performance of different motion plans so that motion planning can be formulated as a variational problem stated in Section 1. Let $\gamma(x, u, t)$ be an optimal trajectory of (14) that connects the state x_0 with x_1 (Figure 2). The trajectory is characterized by a sequence of points $T_0 = t_0 < t_1 < \ldots < t_{N+1} = T_1$ and a sequence of indices $j_0, \ldots, j_N$ such that on the interval $[t_i, t_{i+1}]$, the trajectory belongs to the set D_{j_i} and at the time t_{i+1} it switches from the region D_{j_i} to $D_{j_{i+1}}$. This implies that a solution of the variational problem consists of four components: (a) the number of switches, N; (b) the sequence, $\{D_{j_i}\}_{i=0}^{N}$, of discrete states; (c) the sequence, $\{t_i\}_{i=1}^{N}$, of switching times; (d) the continuous trajectories.

It is difficult to find all four components of the optimal solution (see also [13, 14]). One of the reasons is that in general the optimal solution depends on the sequence of discrete states in very complicated way: even if two sequences are almost identical, trajectories of the continuous state can be very different.

3.2 Method for computing switching times

In tasks such as grasping and walking, the sequence of discrete states is usually known *a priori*. For the grasping task it can be obtained by investigating feasible grasp gaits [15]. In walking, the gait is usually computed in advance to avoid regions that are unsuitable for foot placement [16, 17].

When the sequence of discrete states is given, the motion planning problem reduces to finding the optimal switching times and the optimal trajectories for the continuous state. This problem can be simplified by making the unknown switching times part of the state and introducing a new independent variable with respect to which the switching times are fixed. The resulting variational problem can then be solved using conventional methods.

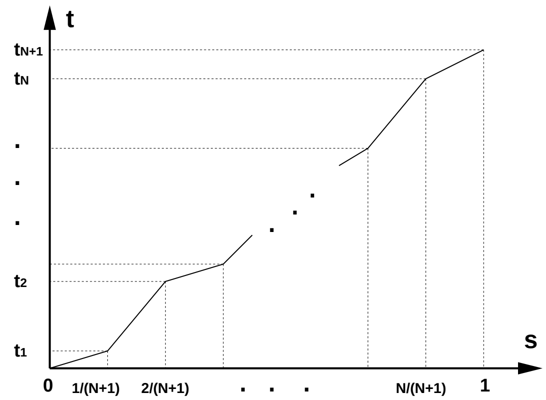

Figure 3: A new independent variable has known values at the switching times t_i.

Assume that the number N and the sequence of discrete states $\{D_{j_i}\}_{i=0}^{N}$ are known. For simplicity, also assume that $T_0 = t_0 = 0$ and $T_1 = t_{N+1} = 1$. We introduce new state variables $x_{n+1}, \ldots, x_{n+N}$ corresponding to the switching times t_i with:

$$x_{n+i} = t_i, \qquad \dot{x}_{n+i} = 0. \tag{16}$$

Subsequently, a new independent variable s is defined. The relation between s and t is linear, but the slope of the curve changes on each interval $[t_i, t_{i+1}]$ (Figure 3). At every chosen fixed point s_i, t equals t_i. For convenience, we set $s_i = i/(N+1)$ so that the following expression is obtained:

$$t = \begin{cases} (N+1)x_{n+1}s, & 0 \leq s \leq \frac{1}{N+1} \\ \ldots & \\ (N+1)(x_{n+i+1} - x_{n+i})s & \\ +(i+1)x_{n+i} - ix_{n+i+1}, & \frac{i}{N+1} < s \leq \frac{i+1}{N+1} \\ \ldots & \\ (N+1)(1 - x_{n+N})s & \\ +(N+1)x_{n+N} - N, & \frac{N}{N+1} < s \leq 1. \end{cases}$$

With the new independent variable, the evolution equation on the interval $[t_i, t_{i+1}]$ becomes:

$$x' = (N+1)(x_{n+i+1} - x_{n+i})F_{j_i}(x, u),$$

where $(.)'$ denotes the derivative of $(.)$ with respect to the new independent variable s. If $\hat{x}$ is the extended state vector:

$$\hat{x} = [x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+N}]^T,$$

a new function can be defined on each interval $\frac{i}{N+1} < s \leq \frac{i+1}{N+1}$:

$$\hat{L}(\hat{x}, u) = (N+1)(x_{n+i+1} - x_{n+i})L(x, u).$$

Finally, the cost functional can be rewritten as:

$$J = \int_0^1 \hat{L}(\hat{x}, u)ds$$

and the task is to minimize J in the extended state space. Points s_i at which the system described by the function $\hat{F}$ switches between the discrete states are known. In the optimal solution, $\hat{x}^*$, the last N components will be the optimal switching times $\{t_i\}_{i=1}^N$ for the original problem.

4 Example

We study an example of two fingers with limited workspace rotating a circular object about a fixed axis in a horizontal plane. In [15], a similar example is used to study grasp gaits. The position of the object is given by its turning angle φ (Figure 4). The center of the object is at the origin of the global coordinate system and the radius of the object is equal to R. Positions of the two fingers in the plane are expressed in polar coordinates and are (r_1, θ_1) and (r_2, θ_2).

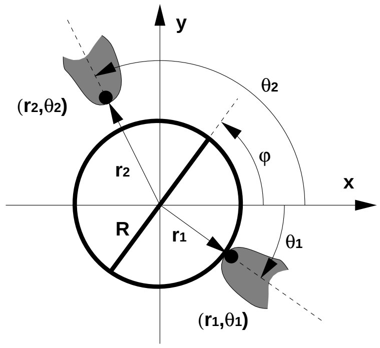

Figure 4: Two fingers rotating a circular object in a plane.

The dynamics of the object is given by:

$$\begin{aligned} I\ddot{\varphi} &= -F_{1x}R\sin\theta_1 + F_{1y}R\cos\theta_1 \\ &\quad - F_{2x}R\sin\theta_2 + F_{2y}R\cos\theta_2, \end{aligned} \quad (17)$$

where I is the moment of inertia of the object around the axis of rotation while F_1 and F_2 are the forces, expressed in the global coordinate frame, that the fingers 1 and 2 exert on the object.

The dynamic equations of the two fingers can be expressed in Cartesian coordinates. For simplicity, we assume that the fingers behave like point masses (located at the finger tip). The dynamics of the finger i is thus given by:

$$\begin{aligned} -F_{ix} + u_{ix} &= m_i(\ddot{r}_i\cos\theta_i - 2\dot{r}_i\dot{\theta}_i\sin\theta_i \\ &\quad - r_i\dot{\theta}_i^2\cos\theta_i - r_i\ddot{\theta}_i\sin\theta_i) \\ -F_{iy} + u_{iy} &= m_i(\ddot{r}_i\sin\theta_i + 2\dot{r}_i\dot{\theta}_i\cos\theta_i \\ &\quad - r_i\dot{\theta}_i^2\sin\theta_i + r_i\ddot{\theta}_i\cos\theta_i), \end{aligned} \quad (18)$$

where m_i is the effective mass of the finger i, and u_i is the 2×1 vector of driving forces for finger i. By defining a state vector:

$$x = [\varphi, \dot{\varphi}, r_1, \theta_1, \dot{r}_1, \dot{\theta}_1, r_2, \theta_2, \dot{r}_2, \dot{\theta}_2]^T \quad (19)$$

the dynamic equations of the object and the two fingers can be transformed into the state-space form.

The workspace of each finger is a cone of angle 2α centered at the origin. The axis of the cone for the first finger corresponds to the half-line $\theta = 0$ while the axis of the cone for the second finger is the half-line $\theta = \pi$. The cones $\mathcal{W}_1$ and $\mathcal{W}_2$ representing the workspace of the fingers 1 and 2, respectively, are thus given by:

$$\begin{aligned} \mathcal{W}_1 &= \{\theta \mid -\alpha \le \theta \le \alpha\} \\ \mathcal{W}_2 &= \{\theta \mid -\alpha + \pi \le \theta \le \alpha + \pi\}. \end{aligned} \quad (20)$$

The state space of the system can be partitioned into three regions:

$$\begin{aligned} D_1 &= \{x \mid \theta_1 \in \mathcal{W}_1, \theta_2 \in \mathcal{W}_2, r_1 > R, r_2 = R\} \\ D_2 &= \{x \mid \theta_1 \in \mathcal{W}_1, \theta_2 \in \mathcal{W}_2, r_1 = R, r_2 > R\} \\ D_3 &= \{x \mid \theta_1 \in \mathcal{W}_1, \theta_2 \in \mathcal{W}_2, r_1 = R, r_2 = R\}. \end{aligned}$$

Region D_1 corresponds to the case when the second finger is in contact with the object while the first finger does not touch the object. Region D_2 describes the opposite situation. In region D_3 both fingers are on the object. For our task we require that exactly one finger is on the object during any finite time interval. This basically reduces the state space to $D_1 \cup D_2$, with the switch between the two corresponding to D_3.

The dynamic equations are the same in all three regions. Therefore, $f = f_1 = f_2$, where f is the system function obtained when the dynamic equations are rewritten in the state space. However, in the region D_1, the constraint $r_1 = 1$ forces $\dot{r}_1 = 0$. Similarly, the requirement $r_2 > 0$ implies $F_2 = 0$. Analogous equations hold for D_2.

Finally, we have to choose the cost functional for the optimal control problem. In robotic tasks, it is often required that the positions and velocities (and in some applications, accelerations and forces) vary "smoothly". In this work, we chose to minimize a measure of the energy necessary to move the two fingers:

$$J = \int_0^1 Ldt = \frac{1}{2}\int_0^1 (u_{1x}^2 + u_{1y}^2 + u_{2x}^2 + u_{2y}^2)dt. \quad (21)$$

This cost functional guarantees continuity of the positions and velocities. Continuous force profiles could be obtained by minimizing the rate of change of forces.

At $t = 0$, the initial conditions are:

$$\begin{aligned} &\varphi = 0, \quad \theta_1 = 0, \quad \theta_2 = \pi, \quad r_1 = R, \quad r_2 = R, \\ &\dot{\varphi} = 0, \quad \dot{\theta}_1 = 0, \quad \dot{\theta}_2 = 0, \quad \dot{r}_1 = 0, \quad \dot{r}_2 = 0. \end{aligned}$$

and we assume that the system immediately passes into the region D_1. We require the object to be rotated through 60°. Both fingers are required to end their motion on the object, but we are not interested where. The final conditions are therefore:

$$\varphi = \frac{\pi}{3}, \quad r_1 = R, \quad r_2 = R,$$
$$\dot{\varphi} = 0, \quad \dot{r}_1 = 0, \quad \dot{r}_2 = 0, \quad \dot{\theta}_1 = 0, \quad \dot{\theta}_2 = 0.$$

In order to use the method from Section 3.2, the number of transitions between the two regions in the state space must be known. Assume that the system switches only once: the object is first rotated with the second finger and the rotation of the object then completed with the first finger. Let t_1 be the time when the system switches from D_1 to D_2. The state vector x is extended with t_1 that satisfies the following state equation:

$$\dot{t}_1 = 0.$$

Next, a new independent variable s is defined:

$$t = \begin{cases} 2t_1 s, & 0 < s \leq 0.5 \\ 2(1-t_1)s + 2t_1 - 1, & 0.5 < s \leq 1.0. \end{cases} \tag{22}$$

The cost function L from Equation (21) becomes:

$$\hat{L}(\hat{x}, u) = \begin{cases} 2t_1 L, & 0 < s \leq 0.5 \\ 2(1-t_1)L, & 0.5 < s \leq 1.0. \end{cases} \tag{23}$$

Similarly, if $f = f_1 = f_2$ is the system function for the independent variable t, the new system functions become:

$$\begin{aligned} \hat{f}_1 &= 2t_1 f \\ \hat{f}_2 &= 2(1-t_1)f. \end{aligned} \tag{24}$$

In each region, it is also necessary to satisfy the following constraints:

$$\begin{aligned} D_1: \quad & r_1 > R \quad r_2 = R \quad \dot{\varphi} = \dot{\theta}_2 \quad F_1 = 0 \\ D_2: \quad & r_1 = R \quad r_2 > R \quad \dot{\varphi} = \dot{\theta}_1 \quad F_2 = 0 \end{aligned} \tag{25}$$

Analogous expressions can be obtained if the fingers change their roles more than once.

The method from Section 2 can now be employed to solve the resulting optimal control problem. One of the advantages of this method is that the adjoint variables and the multipliers are not the unknowns for the optimal control problem, they are only updated after the solution of the optimal control problem has been found. Including state, inputs, adjoint variables and the Lagrange multipliers, there are 38 unknown functions (for a single switch). The unknowns for the variational problem are only the state and input variables which amounts to 19 unknown functions.

4.1 Simulation results

The values of parameters for the simulation were $R = 1m$ and $m_1 = m_2 = 1kg$. The task was to rotate the object for 60° counterclockwise. In Figure 5, the results are shown for the workspace $\alpha = 15°$. In this case, the rotation of the object for 60° can not be achieved with a single switch; the fingers change their roles twice. The value of π is subtracted from θ_2 to compare it with the other two angles. The new independent variable s is shown on the abscissa. The switches between the two regions therefore occur at $s_1 = 0.33$ and $s_2 = 0.67$, which correspond to $t_1 = 3.22s$ and $t_2 = 7.01s$ (computed as part of the motion plan). The figure shows that while one finger rotates the object, the other finger moves towards the lower edge of its workspace to have a wider range of motion once it comes into contact with the object. During the first third of the maneuver, finger 2 rotates the object almost for the entire range of the allowable workspace of 15°. Meanwhile, finger 1 moves to the lower edge of the workspace and subsequently it rotates the object for almost 30° in the second third of the task. While finger 1 is rotating the object, finger 2 moves back towards the middle of its allowable workspace so that it can complete the rotation of the object in the third stage. In the third stage finger 1 stays at the upper edge of its workspace.

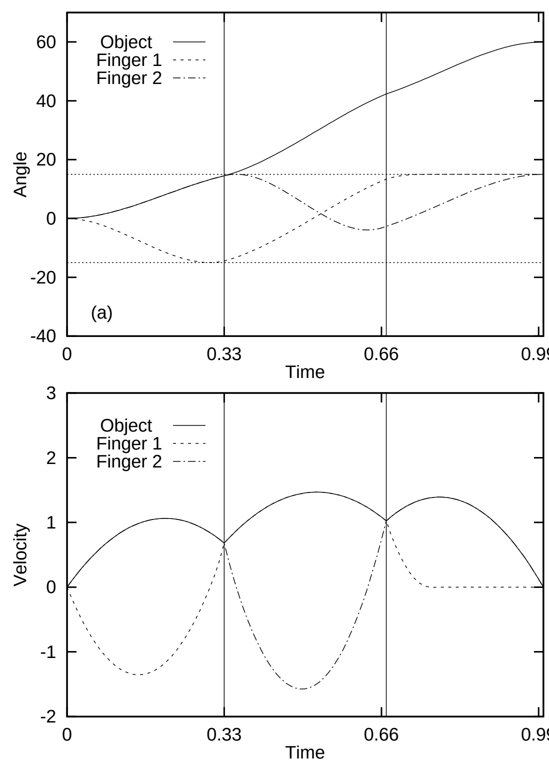

Figure 5: (a) Angles φ, θ_1 and $\theta_2 - \pi$; and (b) velocities $\dot{\varphi}$, $\dot{\theta}_1$ and $\dot{\theta}_2$ for the workspace $\alpha = 15°$.

In this example, the velocities of the contact points on the object and on the finger at the time of establishing or breaking the contact are equal so no impact occurs. Hence, the continuous state is continuous across the switches between discrete states. We also note that the positions of the fingers on the object at the switch are computed as part of the motion plan. In other words, the sequence of discrete states is assumed to be known, but the value of the continuous state at the switches is computed by optimization.

5 Conclusion

We showed that variational calculus can be used as a unified framework for motion planning. In turn, a novel numerical method for solving variational problems was presented. The method uses discretization of a continuous problem provided by finite element analysis and techniques of nonlinear programming to solve the resulting finite-dimensional problem. For motion planning problems in robotics, it is important that the numerical method efficiently handles equality and inequality constraints. We demonstrated that the proposed numerical method satisfies this requirement.

The numerical method was used to compute motion plans for systems that change their dynamic equations at discrete time instants as they move. Such systems can be viewed as having a discrete as well as a continuous state. The existing approaches to motion planning for systems with discrete and continuous state concentrate either on planning the sequence of discrete states to achieve the task or on planning and control of the system within each of the discrete states. In contrast, we addressed both problems at the same level. To find a solution, the sequence of discrete states was assumed to be known in advance, but our approach guarantees that the computed motion plan is compatible with this sequence and the dynamics of the system.

We computed a motion plan for two fingers rotating a circular object in a plane. In this case the (continuous) state space of the system is partitioned into regions (discrete states) that correspond to different grasp configurations. By choosing the cost function appropriately, desired level of smoothness in the velocities, accelerations, and forces at points where the system switches between discrete states was achieved.

Acknowledgment

Support for this research was provided by the NSF grants BCS 92-16691, MSS-91-57156, CISE/CDA 88-22719, ARPA Grant N00014-92-J-1647, NATO grant CRG 911041 and Army Grant DAAH04-96-1-0007. The first author was also supported by a fellowship from the Institute for Research in Cognitive Science at the University of Pennsylvania.

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