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# COORDINATE-FREE FORMULATION OF THE CARTESIAN STIFFNESS MATRIX

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**Abstract.** In the paper we study the Cartesian stiffness matrix using methods of differential geometry. We show that the stiffness of a conservative mechanical system is described by a $\binom{0}{2}$ tensor and that components of the Cartesian stiffness matrix are given by evaluating this tensor on a pair of basis twists. Our formulation leads to three important results: (a) The stiffness matrix does not depend on the parameterization of the manifold; (b) The stiffness matrix depends on the choice of a connection on the manifold; and (c) The standard definition of the Cartesian stiffness matrix assumes an asymmetric connection and this is the reason that the matrix is, in general, asymmetric.

## 1. Introduction

This paper focuses on the static analysis of conservative systems in which the associated potential energy, $\Phi$, is a function of position only. A displacement of a conservative system changes the force which acts on the system. The relation between a displacement and the change in force is given by a stiffness matrix. When the task is described in Cartesian coordinates, the linear and angular velocities of a rigid body are represented by a twist. Accordingly, forces and torques acting on the rigid body are represented as wrenches. Changes in the wrench components as the rigid body moves along the basis twists are given by a $n \times n$ Cartesian stiffness matrix, where $n$ is the dimension of the task space.

It was observed by Griffis and Duffy (1993) and Ciblak and Lipkin (1994) that the Cartesian stiffness matrix associated with a linear elastic coupling is, in general, asymmetric if the forces and moments exerted by the linear elastic coupling are not zero. Pigoski *et al.* (1992) and Ciblak and Lipkin (1994) also showed that the stiffness matrix in the body-fixed reference frame is the transpose of the stiffness matrix in the inertial reference frame. Howard *et al.* (1995) derived these results in differential geometric setting.

In this paper, we study the Cartesian stiffness matrix with the tools of differential geometry. The set of rigid body displacements in $\mathbb{R}^3$ is a Lie group $SE(3)$. We show that an affine connection must be defined on $SE(3)$ in order to compute the stiffness matrix. We define a $\left(\begin{smallmatrix}0\\2\end{smallmatrix}\right)$ stiffness tensor and demonstrate that the components of the stiffness matrix are obtained by evaluating this tensor on a pair of basis twists. This allows us to prove that the stiffness matrix is independent of the choice of parameterization of the manifold $SE(3)$. We demonstrate that by choosing any symmetric connection the stiffness matrix will be always symmetric. Finally, we show that the standard definition (Pigoski *et al.*, 1992; Ciblak and Lipkin, 1994; Howard, 1995) of the stiffness matrix uses a connection which is asymmetric. In general, this results in an asymmetric stiffness tensor and an asymmetric stiffness matrix.

## 2. Kinematics and differential geometry

Consider a rigid body moving in free space. Assume any inertial reference frame {F} fixed in space and a frame {M} fixed to the body at point $O'$ as shown in Figure 1. At each instance, the configuration (position and orientation) of the rigid body can be described by a homogeneous transformation matrix corresponding to the displacement from frame {F} to frame {M}. These transformations form a Lie group $SE(3)$, the special Euclidean group in three-dimensions (Murray *et al.*, 1994).

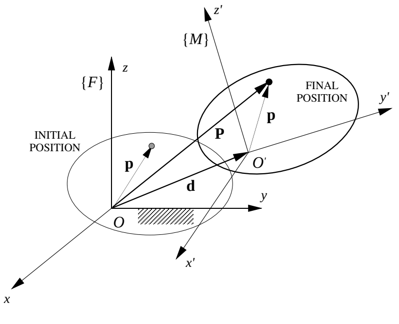

*Figure 1.* The inertial (fixed) frame and the moving frame attached to the rigid body.

On a Lie group, the tangent space at the group identity has the structure of a Lie algebra. The Lie algebra of $SE(3)$ is denoted by $se(3)$ and is given by:

$$se(3) = \left\{ \begin{bmatrix} \Omega & v \\ 0 & 0 \end{bmatrix}, \Omega \in \mathrm{IR}^{3\times 3}, v \in \mathrm{IR}^3, \Omega^T = -\Omega \right\}. \tag{1}$$

A $3 \times 3$ skew-symmetric matrix $\Omega$ can be uniquely identified with a vector $\omega \in \mathrm{IR}^3$ so that for an arbitrary vector $x \in \mathrm{IR}^3$, $\Omega x = \omega \times x$, where $\times$ is the cross product in $\mathrm{IR}^3$. Each element $T \in se(3)$ can be thus identified with a vector pair $\{\omega, v\}$.

Since $se(3)$ is a vector space, any element can be expressed as a $6 \times 1$ vector of components corresponding to a chosen basis. The standard basis that will be used throughout the paper is:

$$\begin{aligned} L_1 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad L_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad L_3 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ L_4 &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad L_5 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad L_6 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{2}$$

This basis has the property that the components of an element $T \in se(3)$ are given precisely by the vector pair $\{\omega, v\}$ mentioned above.

The product operation on a Lie algebra is called a Lie bracket. The Lie bracket of two elements $T_1, T_2 \in se(3)$ is defined by:

$$[T_1, T_2] = T_1 T_2 - T_2 T_1. \tag{3}$$

The coefficients $C_{ij}^k$ corresponding to the Lie brackets of the basis vectors:

$$[L_i, L_j] = \sum_k C_{ij}^k L_k, \tag{4}$$

are called *structure constants* of the Lie algebra (Sattinger and Weaver, 1986). The nonzero structure constants for the basis (2) are:

$$\begin{aligned} C_{12}^3 = C_{31}^2 = C_{23}^1 = C_{15}^6 = C_{26}^4 = C_{34}^5 = C_{42}^6 = C_{53}^4 = C_{61}^5 = 1 \\ C_{21}^3 = C_{13}^2 = C_{32}^1 = C_{51}^6 = C_{62}^4 = C_{43}^5 = C_{24}^6 = C_{35}^4 = C_{16}^5 = -1. \end{aligned} \tag{5}$$

### 2.1. RIGHT INVARIANT VECTOR FIELDS

A differentiable vector field on a manifold is a smooth assignment of a tangent vector to each element of the manifold. At each point, a vector field defines a unique *integral curve* to which it is tangent (do Carmo, 1992). Formally, a vector field $X$ is a derivation operator which, given a real-valued differentiable function $f$, returns its derivative along the integral curves of $X$.

On $SE(3)$, an example of a differentiable vector field, $X$, is obtained by setting:

$$X(A) = \hat{T}(A) = TA, \tag{6}$$

where $T \in se(3)$ and $A \in SE(3)$. Such a vector field is called *a right invariant vector field* and we use the notation $\hat{T}$ to indicate that the vector field was obtained by the right translation of the Lie algebra element $T$. By construction, the space of right invariant vector fields is isomorphic to the Lie algebra $se(3)$. In particular, $[\hat{L}_i, \hat{L}_j] = \widehat{[L_i, L_j]} = \sum_k C_{ij}^k \hat{L}_k$ (do Carmo, 1992).

Since the vectors $L_1, L_2, \ldots, L_6$ are a basis for the Lie algebra $se(3)$, the vectors $\hat{L}_1(A), \ldots, \hat{L}_6(A)$ form a basis of the tangent space at any point $A \in SE(3)$. Therefore, any vector field $X$ can be expressed as $X = \sum_{i=1}^6 X^i \hat{L}_i$, where the coefficients $X^i$ are real-valued functions. By defining $\omega = [X^1, X^2, X^3]^T$ and $v = [X^4, X^5, X^6]^T$, we can associate a vector pair $\{\omega, v\}$ to an arbitrary vector field $X$.

### 2.2. VELOCITY OF A RIGID BODY

Take a rigid body moving in space. The motion of the rigid body can be described by a curve $A(t) : \mathbb{R} \rightarrow SE(3)$. The tangent vector to this curve, $\frac{dA}{dt}$, is the velocity of the rigid body. The tangent vector $\frac{dA}{dt}$ can be mapped to an element $T$ of the Lie algebra $se(3)$ by:

$$T = \dot{A}A^{-1} = \begin{bmatrix} \dot{R}R^T & -\dot{R}R^T d + \dot{d} \\ 0 & 0 \end{bmatrix}. \tag{7}$$

It can be shown that $T$ does not depend on the choice of the body-fixed frame {M} and is called the right invariant representation of the velocity. If $T$ is expressed in the basis (2), the vector of components $\{\omega, v\}$ corresponds to the twist describing the instantaneous velocity of the rigid body. More precisely, $\omega$ is the angular velocity of the rigid body while $v$ is the linear velocity of the point on the rigid body that is instantaneously at the origin $O$ of the frame {F}, both expressed in the inertial frame {F}. The Lie algebra $se(3)$ thus represents the space of twists. In this space, the basis vectors $L_1$, $L_2$ and $L_3$ correspond to instantaneous rotations about and $L_4$, $L_5$ and $L_6$ to instantaneous translations along the Cartesian axes $x$, $y$ and $z$, respectively.

Equation (7) shows that:

$$\dot{A} = T\,A = \left(\sum_{i=1}^{6} T^i L_i\right) A = \sum_{i=1}^{6} T^i (L_i A) = \sum_{i=1}^{6} T^i \hat{L}_i. \tag{8}$$

We conclude that if the velocity of the rigid body is expressed in the basis $\hat{L}_1, \ldots, \hat{L}_6$, the components are equal to the components of the twist describing the instantaneous velocity. We thus call the basis vector fields $\hat{L}_1, \ldots, \hat{L}_6$ *the basis twists.*

## 3. Static analysis on $SE(3)$

### 3.1. TWISTS AND WRENCHES

If a wrench $W = \{\tau, F\}$ acts on the rigid body during a displacement $\Delta T = \{\omega, v\}\Delta t$, it produces work, $\Delta E = (F^T v + \tau^T \omega)\Delta t$, which is a scalar. Wrenches therefore belong to the dual of the space of twists, $W \in se^*(3)$. This implies that a force[1] acting on the rigid body that moves in space represents a one-form (Simo, 1992).

Given a basis for the vector fields, there exists a natural basis for one-forms, called *the dual basis.* If $\{\hat{L}_i\}$ is a basis for the vector fields, the dual basis for the one-forms, $\{\hat{\lambda}^i\}$, satisfies:

$$\ll \hat{\lambda}^i \, ; \, \hat{L}_j \gg = \delta^i_j, \tag{9}$$

where $\ll \hat{\lambda}^i \, ; \, \hat{L}_j \gg$ represents the action of a one-form $\hat{\lambda}^i$ on a vector field $\hat{L}_j$ and $\delta^i_j$ is the Kronecker $\delta$. If a one-form $\mathcal{F} = \mathcal{F}_i \hat{\lambda}^i$ is expressed in the dual basis, it is easy to see that its action on a vector field $V = V^j \hat{L}_j$ is given by $\ll \mathcal{F} \, ; \, V \gg = \mathcal{F}_i V^i$. Here and in the rest of the chapter we use the Einstein summation convention.

It is not difficult to see that the components of the force one-form in the basis $\{\hat{\lambda}^i\}$ that is dual to the basis twists $\{\hat{L}_i\}$, are exactly the components of the corresponding wrench in $se^*(3)$.

### 3.2. FORCES IN A POTENTIAL FIELD

We are interested in the static analysis of conservative systems in which the associated potential field, $\Phi$, is a function of position only. In $\mathbb{R}^3$, the force generated by a potential field $\phi$ is equal to the negative gradient of the potential field, $F = -\mathrm{grad}(\phi)$. This can be generalized to an arbitrary manifold if we generalize the notion of gradient (Schutz, 1980). A gradient of a real-valued function $f$, denoted by $df$, is a one-form, whose action on an arbitrary vector field $X$ is defined by:

$$\ll df \, ; \, X \gg = X(f). \tag{10}$$

Here we remind the reader that a vector field on a manifold must be regarded as a derivation (see Section 2.1).

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[1] By force we mean the generalized force consisting of both, forces and torques acting on the rigid body.

The force one-form, $\mathcal{F}$, corresponding to a potential field $\Phi$ is therefore:

$$\mathcal{F} = -d\Phi. \tag{11}$$

To obtain the wrench that corresponds to the force one-form $\mathcal{F}$ at a point $A \in SE(3)$, the one form must be expressed in the basis dual to the basis twists. We can see from Eq. (9) that the components of the wrench are given by $W_i = \ll \mathcal{F}\,; \hat{L}_i \gg = -\hat{L}_i(\Phi)$ and that $\mathcal{F} = W_i\,\hat{\lambda}^i$.

### 3.3. COVARIANT DERIVATIVES

The Cartesian stiffness matrix describes how the components of a wrench change in the directions given by the basis twists. This suggests that the wrench has to be differentiated along the basis twists. But a wrench belongs to the space $se^*(3)$, the dual of the space of twists, so it can not be differentiated along a vector field, which is defined over the entire manifold. The quantity that can be differentiated with respect to a vector field is the force one-form $\mathcal{F}$.

Differentiation of vector fields and one forms on the manifold is not defined unless the manifold is endowed with *an affine connection.* Given a curve $A(t)$ on the manifold, the affine connection specifies how an element $X$ of the tangent space at point $A(t_1)$ can be mapped to an element $X'$ of the tangent space at some other point $A(t_2)$. The vector $X'$ is called the parallel transport of $X$ along $A(t)$. In this way, we can define a *covariant derivative* of a vector field $X(t)$ along a curve $A(t)$ by:

$$\left.\frac{DX}{dt}\right|_{t_0} = \lim_{h\to 0} \frac{X^{t_0}(t_0+h) - X(t_0)}{t}, \tag{12}$$

where $X^{t_0}(t_0+h)$ is the parallel transport of the vector $X(A(t_0+h))$ along $A(t)$ to the point $A(t_0)$. By taking covariant derivatives of a vector field $X$ along the integral curves of another vector field $Y$, we obtain a covariant derivative, $\nabla_Y X$, of the vector field $X$ with respect to the vector field $Y$:

$$\nabla_Y X|_{A_0} = \left.\frac{DX}{dt}\right|_{t_0}, \tag{13}$$

where $\frac{DX}{dt}$ is taken along the integral curve of $Y$ passing through $A_0$ at $t = t_0$. The coefficients $\Gamma^k_{ji}$ of the covariant derivative of a basis vector field along another basis vector field,

$$\nabla_{\hat{L}_i} \hat{L}_j = \Gamma^k_{ji} \hat{L}_k, \tag{14}$$

are called *Christoffel symbols*[2].

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[2] Some texts (do Carmo, 1992) reserve the term only for the coordinate basis vectors. We follow the more general definition from (Schutz, 1980) in which the basis vectors can be arbitrary.

**Remark 3.1** If for all vector fields $X$ and $Y$, a connection satisfies:

$$\nabla_X Y - \nabla_Y X = [X, Y], \tag{15}$$

the connection is said to be *symmetric.*

A covariant derivative of a one-form can be defined through the covariant derivative of a vector field. Loosely speaking, to obtain a covariant derivative $\nabla_U \mathcal{F}$ of a one-form $\mathcal{F}$ along a vector field $U$, we use a "generalization" of the Leibniz' rule (Schutz, 1980). Let $\mathcal{F}$ be a one-form and $U$ and $V$ vector fields. We then have:

$$\nabla_U \ll \mathcal{F}\,;\, V \gg = \ll \nabla_U \mathcal{F}\,;\, V \gg + \ll \mathcal{F}\,;\, \nabla_U V \gg . \tag{16}$$

Since $\ll \mathcal{F}\,;\, V \gg$ is a real-valued function, $\nabla_U \ll \mathcal{F}\,;\, V \gg = U(\ll \mathcal{F}\,;\, V \gg)$. The rule for a covariant derivative of $\mathcal{F}$ along $U$ is therefore:

$$\ll \nabla_U \mathcal{F}\,;\, V \gg = U(\ll \mathcal{F}\,;\, V \gg) - \ll \mathcal{F}\,;\, \nabla_U V \gg . \tag{17}$$

### 3.4. THE CARTESIAN STIFFNESS MATRIX AND STIFFNESS TENSOR

The Cartesian stiffness matrix is obtained by differentiating the force one-form in the directions of the basis twists. Since we are interested in changes of the wrench components in the directions of the basis twists, the resulting one-forms $\nabla_{\hat{L}_i} d\Phi$ must be expressed in the basis dual to the basis twists. This leads to the following definition for the coefficients of the stiffness matrix:

$$K_{ij} = \ll \nabla_{\hat{L}_j} d\Phi\,;\, \hat{L}_i \gg . \tag{18}$$

(The minus sign is omitted to conform to the usual definition of the stiffness matrix in the literature.) Equation (18) can be expanded using Eq. (17):

$$\begin{aligned} K_{ij} &= \ll \nabla_{\hat{L}_j} d\Phi\,;\, \hat{L}_i \gg \\ &= \hat{L}_j(\ll d\Phi\,;\, \hat{L}_i \gg) - \ll d\Phi\,;\, \nabla_{\hat{L}_j} \hat{L}_i \gg = (\hat{L}_j \hat{L}_i - \nabla_{\hat{L}_j} \hat{L}_i)(\Phi). \end{aligned} \tag{19}$$

A closer look at Eq. (18) reveals that the right-hand side is linear in both vector components:

1. $\ll \nabla_{f_j \hat{L}_j + f_k \hat{L}_k} d\Phi\,;\, \hat{L}_i \gg = f_j \ll \nabla_{\hat{L}_j} d\Phi\,;\, \hat{L}_i \gg + f_k \ll \nabla_{\hat{L}_k} d\Phi\,;\, \hat{L}_i \gg$ (linearity of the covariant derivative).
2. $\ll \nabla_{\hat{L}_j} d\Phi\,;\, f_i \hat{L}_i + f_k \hat{L}_k \gg = f_i \ll \nabla_{\hat{L}_j} d\Phi\,;\, \hat{L}_i \gg + f_k \ll \nabla_{\hat{L}_j} d\Phi\,;\, \hat{L}_k \gg$ (linearity of the one-form).

This suggests introducing a $\binom{0}{2}$ *stiffness tensor* $K = \nabla d\Phi$ defined for arbitrary vector fields $X$ and $Y$ by:

$$K(X, Y) = \ll \nabla_Y d\Phi\,;\, X \gg . \tag{20}$$

Equation (18) thus becomes:

$$K_{ij} = K(\hat{L}_i, \hat{L}_j). \tag{21}$$

The last equation leads to the following propositions.

**Proposition 3.2** *The component $K_{ij}$ of the Cartesian stiffness matrix is obtained by evaluating the stiffness tensor $K = \nabla d\Phi$ on the pair of basis twists $\hat{L}_i$ and $\hat{L}_j$.*

**Proposition 3.3** *The Cartesian stiffness matrix does not depend on the choice of the coordinates for $SE(3)$ (parameterization of the space).*

***Proof:*** Equation (21) suggests that the entries of the stiffness matrix depend on the $\binom{0}{2}$ tensor $K$ and on the basis twists $\{\hat{L}_i\}$. By definition, the tensor $K$ is independent of the choice of the coordinates (this makes it a tensor). Further, the basis twists are defined by the right translation of the canonical basis of the Lie algebra $se(3)$ and this operation is again coordinate invariant.

□

**Remark 3.4** The independence of the Cartesian stiffness matrix from the parameterization of $SE(3)$ is also demonstrated by the fact that we never introduced the coordinates for $SE(3)$!

**Proposition 3.5** *If the affine connection used in the definition of the stiffness tensor $K$ is symmetric, then the stiffness tensor is symmetric.*

***Proof:*** For a symmetric connection (Eq. 15), we obtain:

$$\begin{aligned} K(X,Y) &= (XY - \nabla_X Y)(\Phi) \qquad (22)\\ &= (XY - [X,Y] - \nabla_Y X)(\Phi) = (YX - \nabla_Y X)(\Phi) = K(Y,X). \end{aligned}$$

□

**Corollary 3.6** *If the connection used to define the stiffness tensor in Eq. (20) is symmetric the Cartesian stiffness matrix is always symmetric.*

### 3.5. STANDARD DEFINITION OF THE CARTESIAN STIFFNESS MATRIX

In this section we compare our coordinate free definition of the Cartesian stiffness matrix based on the stiffness tensor with the definition that is usually found in the literature (Pigoski *et al.*, 1992; Ciblak and Lipkin, 1994; Howard, 1995). In these works, the coefficient of the stiffness matrix $K_{ij}$ is computed by taking a small displacement $\Delta T^j$ in the direction of the basis twist $\hat{L}_j$, computing the corresponding change of the wrench component $W_i$ and taking the limit of the quotient of the two as the displacement goes to 0:

$$K_{ij} = -\lim_{\Delta T^j \to 0} \frac{\Delta W_i}{\Delta T^j}. \tag{23}$$

We observe that this corresponds precisely to derivative of $W_i$ along the integral curve of $\hat{L}_j$:

$$K_{ij} = -\hat{L}_j(W_i). \tag{24}$$

But $W_i = - \ll d\Phi\, ;\, \hat{L}_i \gg = -\hat{L}_i(\Phi)$, so the expression for the element $K_{ij}$ becomes:

$$K_{ij} = \hat{L}_j \hat{L}_i(\Phi). \tag{25}$$

If we compare Eq. (25) with Eq. (20), it is immediately apparent that the term with the connection is missing in (25). Hence, we conclude that the standard definition of the stiffness matrix requires:

$$\nabla_{\hat{L}_i} \hat{L}_j = 0 \quad \forall i, j. \tag{26}$$

This equation completely specifies the connection $\nabla$. If the vector fields are expressed with the basis twists $\{\hat{L}_i\}$, the Christoffel symbols for this connection all vanish:

$$\Gamma^k_{ji} = 0 \quad \forall i, j, k. \tag{27}$$

This result immediately implies that the connection (26) is not symmetric:

$$0 = \nabla_{\hat{L}_i} \hat{L}_j - \nabla_{\hat{L}_j} \hat{L}_i \neq [\hat{L}_i, \hat{L}_j]. \tag{28}$$

(See Eq. (6) for the values of the Lie brackets $[\hat{L}_i, \hat{L}_j]$.)

A consequence of this fact is stated in the following proposition (Ciblak and Lipkin, 1994; Howard *et al.*, 1995).

**Proposition 3.7** *The Cartesian stiffness matrix, as defined in (*Pigoski *et al.*, 1992; Ciblak and Lipkin, 1994; Howard, 1995*), is in general asymmetric.*

***Proof:*** This statement is the consequence of the asymmetry of the connection (see Proposition 3.5). From Eq. (25) we compute:

$$K_{ij} - K_{ji} = [\hat{L}_j, \hat{L}_i](\Phi). \tag{29}$$

Since the basis twists are not the coordinate basis, the Lie brackets do not vanish. It is also easy to see that at stationary points of the potential field $\Phi$, the stiffness matrix becomes symmetric (at a stationary point, $X(\Phi) = 0$ for an arbitrary vector field $X$).

□

## 4. Conclusion

We presented a coordinate-free formulation of the Cartesian stiffness matrix for conservative mechanical systems in which the potential field $\Phi$ is a function of position only. We chose a basis for the vector fields so that in this basis the components of the velocity vector field are equal to the components of the twist associated with the motion of the rigid body. We argued that the force generated by the potential field $\Phi$ is a one-form. Further, in the basis dual to the basis chosen for the vector fields, the components of this one-form are equal to the components of the associated wrench. We demonstrated that an affine connection must be introduced to the manifold $SE(3)$ in order to compute the stiffness matrix and observed that the components of the stiffness matrix are obtained by evaluating a $\binom{0}{2}$ tensor on a pair of basis twists. This enabled us to prove that the stiffness matrix is independent of the choice of parameterization of the space (the manifold $SE(3)$). We demonstrated that by choosing a symmetric connection, the stiffness matrix is always symmetric. Finally, we showed that the standard definition of the Cartesian stiffness matrix assumes a connection which is asymmetric. In general, this results in an asymmetric stiffness tensor and an asymmetric stiffness matrix.

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