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Motion Planning for a Class of Hybrid Locomotion Systems

Miloš Žefran
Electrical Engineering and Computer Science
University of Illinois at Chicago
Chicago, IL 60607
mzefran@eecs.uic.edu

Abstract

A mechanical system subject to constraints typically undergoes changes in the dynamics as the set of active constraints changes, and can be thus modeled as a hybrid system. The focus of this paper is on devices that switch between constraints in order to locomote. An important component of a motion plan for a hybrid system is a sequence of dynamic regimes that it needs to traverse to reach the goal. Finding this sequence can lead to a combinatorial complexity of motion planning algorithms. We show that controllability analysis can be exploited to reduce this complexity. However, since switches between dynamic regimes result in discontinuous changes in the vector fields describing the system, classical controllability results are not applicable. We characterize the controllability of locomotion devices that are able to switch between dynamic regimes at an arbitrary point in the configuration space. We use the analysis to compute sub-optimal motion plans for a simple locomotion device consisting of a planar linkage sliding on a frictionless surface that can clamp a subset of its links to the surface.

1 Introduction

In this paper we study motion planning for a class of systems that can locomote by switching between constraints. Locomotion devices belong to the class of mechanical control systems (Lagrangian systems) and have a special structure that can be fruitfully exploited for control. However, several locomotion modalities, including legged locomotion involve impacts and switches in the dynamic behavior and are therefore inherently non-smooth. Non-smooth locomotion systems belong to the class of hybrid systems. While a number of approaches to modeling and control of hybrid systems exist [1, 6], these works are in general not applicable to locomotion. Some exceptions are [5] which provides a quasi-static controllability analysis of locomotion devices, and [15] where a method for stabilizing systems with changing dynamics is described.

The state space of a non-smooth locomotion system can be partitioned into regions D_j so that in the region D_j the system is described with a set of differential equations $\dot{x} = F_j(x, u)$. The regions D_j are called discrete states; within each discrete state, the differential equations describe the evolution of the continuous state. Let $\gamma(x, u, t)$ be a trajectory that connects the state x_0 with x_1 (Figure 1). This trajectory is characterized by a sequence of time instants $T_0 = t_0 < t_1 < \ldots < t_{N+1} = T_1$ and a sequence of indices $j_0, \ldots, j_N$ such that on the interval $[t_i, t_{i+1}]$, the trajectory belongs to the set D_{j_i} and

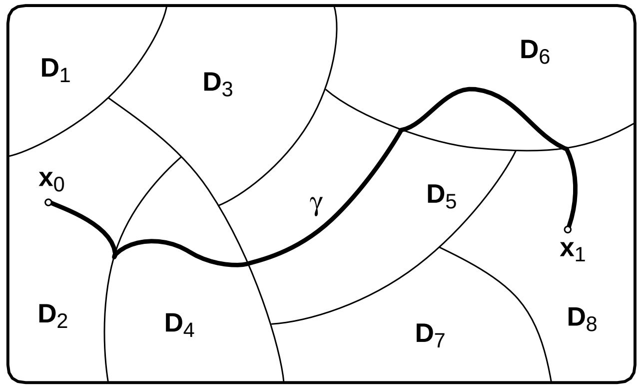

Figure 1: A trajectory of a system with discrete and continuous state.

at the time t_{i+1} it switches from the region D_{j_i} to $D_{j_{i+1}}$. A motion plan for a hybrid locomotion system thus consists of three components:

(i) A sequence, $\{D_{j_i}\}_{i=0}^{N}$, of discrete states.

(ii) A sequence, $\{t_i\}_{i=1}^{N}$, of switching times.

(iii) Continuous trajectories on each interval $[t_i, t_{i+1}]$.

We have shown in our previous work [16] that if the sequence of discrete states is known, motion plans can be efficiently computed using variational approaches. However, finding a sequence of discrete states involves descending the graph describing the discrete evolution of the system and can easily lead to a combinatorial explosion. In [14] we showed that for a certain class of hybrid locomotion systems, the sequence of discrete states necessary to move in a particular direction can be inferred from the controllability analysis of the system. The aim of the present paper is therefore to put together these two sets of results and demonstrate how the analytical tools from [14] can be used in numerical motion planning algorithms such as [16].

The main assumption that allows us to perform the controllability analysis is that the system can switch between constraints at any point in the configuration space. In other words, the distribution that describes the set of admissible directions for a particular constraint is defined at every point. An example is a planar linkage sliding on a frictionless surface that can arbitrarily clamp a subset of its links to the surface. Note that the above assumption does not hold for legged robots; there, the switch to a constrained regime only takes place when a leg hits the ground (the constrained distribution is only defined on a submanifold).

The underlying mathematical framework of this paper is Riemannian geometry as it applies to mechanical control systems [11]. Modeling and controllability results for smooth mechanical systems are discussed in [10]. Of special importance in the controllability analysis are series describing the evolution of the system [12, 13]. Such series are also a basis of several motion planning algorithms [2, 7, 8].

We base the paper on the modeling framework for constrained mechanical systems outlined in [9]. We define a class of hybrid mechanical control systems and show that under the appropriate assumptions the controllability results from [9] can be extended to this class of systems. Finally, we show how the controllability analysis can be used in motion planning.

2 Mechanical control systems

Let Q be the configuration manifold of the system with coordinates $q = (q_1, \dots, q_n)$. At every $q \in Q$, the kinetic energy of the system defines a Riemannian metric, M_q. Hamilton's principle states that unforced motions of the system correspond to geodesics with respect to the metric M[1] and are thus given by the geodesic equation:

$$\nabla_{\dot{q}}\dot{q} = 0 \tag{1}$$

where ∇ is the *Levi-Civita connection* corresponding to M [4]. In coordinates (we use the summation convention throughout the paper), an affine connection is given by:

$$\nabla_X Y = \left(\frac{\partial Y^i}{\partial q^j} X^j + \Gamma^i_{jk} X^j Y^k \right) \frac{\partial}{\partial q^i} \tag{2}$$

where the coefficients Γ^i_{jk} are known as Christoffel symbols.

When an external force (a one-form) F acts on the system, the dynamic equations take the form:

$$\nabla_{\dot{q}}\dot{q} = Y \tag{3}$$

where $Y(q) = M_q^{-1}F(q)$ is now a vector field. Written in coordinates, these *forced Euler-Lagrange* equations take the familiar form:

$$\ddot{q} + M_q^{-1}C(q, \dot{q}) = M_q^{-1}F \tag{4}$$

where M_q is the inertia matrix and $C(q, \dot{q})$ are the Coriolis and centrifugal forces. Formally a *mechanical control system* can be defined as a tuple $(Q, M, \mathcal{F}, U)$, where Q is an n-dimensional configuration manifold, M is a Riemannian metric on Q (the kinetic energy), $\mathcal{F} = \text{span}\{F^1, \dots, F^m\}$ is the input co-distribution (input forces), and $U \subset \mathbb{R}^m$ is the set of inputs. Later in the paper we will also need the notion of input distribution, $\mathcal{Y} = \text{span}\{M^{-1}F^i\}$.

2.1 Constraints

From the control point of view, constraints on a mechanical system limit the set of directions in which the system can move. Therefore, an intrinsic description of a constraint is a distribution on Q, describing at each point the set of feasible velocities. Such description applies both to holonomic and nonholonomic constraints.

A mechanical control system together with a constrained distribution will be called a *constrained mechanical control system*, $\Sigma = (Q, M, \mathcal{F}, \mathcal{D}, U)$. It was shown in [9] that the dynamic equations for such a system can be also written using an affine connection:

$$\widetilde{\nabla}_{\dot{q}}\dot{q} = u^k P_{\mathcal{D}}(Y^k), \tag{5}$$

where $P_{\mathcal{D}} : TQ \to TQ$ is the orthogonal projection to $\mathcal{D}$ (with respect to the metric M) and $\widetilde{\nabla}$ is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + A^{-1} \left(\nabla_X A(I - P) \right)(Y), \qquad \forall X, Y$$

[1]The controllability analysis is simpler if the potential forces are present so we will neglect them in this work.

where A is an invertible $(1,1)$ tensor. For different choices of A we will get different affine connections, however it can be easily checked that if Y is a vector field which takes values in D then all these connections yield the same value. This implies that each of these connections yields the same dynamic equations on the constraint distribution $\mathcal{D}$. The tensor A is usually chosen so that the computations are simplified.

The importance of defining the connection $\widetilde{\nabla}$ is that equation (5) is formally identical to equation (3). Such description therefore provides a unified treatment of both constrained and unconstrained mechanical control systems, see [9].

2.2 Hybrid mechanical control systems

A *hybrid mechanical control system* consists of a mechanical control system $(Q, M, \mathcal{F}, U)$ together with a given set of constraint distributions $\mathcal{D}_i$, where i belongs to an index set I. Each constraint $\mathcal{D}_i$ yields a constrained mechanical control system $\Sigma_i = (Q, M, \mathcal{F}, U, \mathcal{D}_i)$, with associated affine connection ∇_i and input distribution $\mathcal{Y}_i = \text{span}\{P_{\mathcal{D}_i} M^{-1} F\}$. Formally, the hybrid mechanical control system is therefore a tuple $(I, Q, M, \mathcal{F}, U, \{\mathcal{D}_i\}_{i \in I})$. Slightly more general definition can be found in [3].

The evolution of a hybrid mechanical control system can be described as follows. The system starts in a state $((q, \dot{q}), i) \in TQ \times I$ and it evolves according to the dynamics given by ∇_i and the chosen set of controls. At any point, we can choose to switch to any other discrete state. Whenever the system switches between two discrete states i and j (constraint distributions $\mathcal{D}_i$ and $\mathcal{D}_j$) it undergoes impact. The velocity after the impact is the orthogonal projection onto $\mathcal{D}_j$ of the velocity before the impact:

$$\dot{q}(t^+) = P_{\mathcal{D}_j} \dot{q}(t^-) \tag{6}$$

3 Equilibrium controllability

In the case of mechanical control systems two controllability questions can be asked: (a) what points in the *phase space* (i.e., configuration and velocity) can be reached; and (b) what *configurations* can be reached. The first question can be addressed using standard nonlinear controllability methods. We will show that in the case of hybrid mechanical control systems, the second question is particularly interesting. For smooth systems, the question was posed and answered in [10] and we briefly review this work here.

Consider a system described by (5). Let q_0 be a point in Q and let W be a neighborhood of q_0. The *reachable set of q_0 within W* is

$$\mathcal{R}_Q^W(q_0, \leq T) =$$
$$\cup_{\tau \leq T} \{x \in Q \mid \exists \text{ a solution } q(t) \text{ to (5) s.t. } \dot{q}(0) = 0,\ q(t) \in W \text{ for } t \in [0, \tau],\ q(\tau) = x\}.$$

Note that the definition of $\mathcal{R}_Q^W(q_0, \leq T)$ only involves the configurations, not the velocities. The system (5) is *locally configuration controllable* at q_0 if there exists a time T such that $\mathcal{R}_Q^W(q_0, \leq T)$ contains a neighborhood of q_0 for any neighborhood W of q_0, and *equilibrium controllable* on $W \subset Q$, if for any two equilibrium points $q_1, q_2 \in W$, there exists an input $\{u^k(t), t \in [0, T]\}$ and a solution $\{q(t), t \in [0, T]\}$ such that $q(0) = q_1$, $q(T) = q_2$, $q(t) \in W$ for all $t \in [0, T]$, and $\dot{q}(0) = 0$, $\dot{q}(T) = 0$.

To characterize the configuration controllability, we introduce the following operations. As in [10], we define the *symmetric product* of two vector fields as:

$$\langle X : Y \rangle = \nabla_X Y + \nabla_Y X.$$

Let $\mathcal{X} = \{X_1, \dots, X_m\}$ be a family of vector fields. We define $\mathrm{Lie}(\mathcal{X})$ to be the closure of $\mathcal{X}$ under the Lie bracket operation (the involutive closure), and we let $\mathrm{Sym}(\mathcal{X})$ be the closure of $\mathcal{X}$ under the symmetric product operation. Within the set $\mathrm{Sym}(\mathcal{X})$, we define the *order* of a symmetric product to be the number of vector fields X_j present in it. We say that a symmetric product is *bad* if it contains an even number of each X_i. Otherwise the product is said to be *good*. The controllability tests are then:

Theorem 3.1 ([10]). *The system is configuration controllable at q_0 if:*

(i) the rank of $\mathrm{Lie}(\mathrm{Sym}(\mathcal{Y}))$ *is full;*

(ii) at q_0, every bad symmetric product is a linear combination of lower order good symmetric products.

If these conditions are verified at every $q \in W$, then the system is equilibrium controllable on W.

3.1 Hybrid mechanical control systems

In [3] we studied configuration and equilibrium controllability for hybrid mechanical control systems. Since the analysis above requires computation of the brackets at zero velocity, we had to restrict our analysis to the case when the switches between different regimes occur at zero velocity. We showed:

Proposition 3.2 ([3]). *A hybrid mechanical control system is equilibrium controllable on an open set W if the following two conditions hold:*

(i) in each discrete state i, every bad symmetric product is a linear combination of lower order good symmetric products

(ii) the rank of $\mathrm{Lie}(\sum_{i\in I} \mathrm{Sym}_i(\mathcal{Y}_i))(q)$ *is full for all $q \in W$.*

Restricting a hybrid mechanical control system to switches at zero velocity is overly restrictive. It was shown in [14] that it is possible to generalize the above result to switches at nonzero velocity. The key observation is that if in some regime i the system can generate a nonzero velocity at q_0 (by making a small loop in the configuration space), then we can use the nonzero velocity to exploit directions generated by the brackets that would otherwise vanish. These additional directions can not be generated by starting at zero velocity and must be evaluated for those values of v that can be generated by making small loops from q_0. These directions are characterized by $\mathrm{Sym}(\mathcal{Y})$, see [10].

For simplicity we assume that the system only has two regimes (the system can switch between free motion and motion in directions $\mathcal{D}$), $\Sigma = (\{1,2\}, Q, M, \mathcal{F}, U, \{TQ, \mathcal{D}\})$.

Proposition 3.3 ([14]). *The system Σ is equilibrium controllable on some neighborhood W of q_0 if the following conditions hold:*

(i) $\mathcal{D} \subseteq \mathrm{span}\,\{\mathrm{Sym}(\{Y_i^1\}) \cup \mathrm{Sym}(\{Y_i^2\})\}$*, where $Y_i^1 = Y_i$ and $Y_i^2 = P_{\mathcal{D}} Y_i^1$.*

(ii) For $k = 1, 2$, the bad symmetric products $\langle Y_i^k : Y_i^k \rangle$ are spanned by the vector fields Y_i^k.

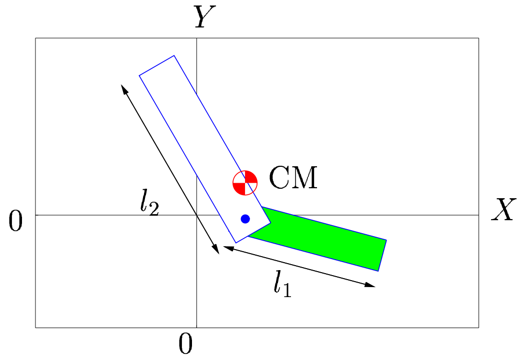

Figure 2: Sliding and clamping mechanism with two links. One of the links can be clamped to the floor so that it is completely immobilized.

(iii) For every $q \in W$:

$$TQ = \text{span}\left\{\mathcal{D} \cup \{Y_i^k, [Y_i^k, Y_j^l]\} \cup \{\nabla_v Y_i^k, \nabla^2 Y_i^k(v,v) + R(v, Y_i^k)v\}_{v \in \text{Sym}(\{Y_i^l\})}\right\}$$

where k and l must be different regimes.

Condition (ii) corresponds to condition (i) in Proposition 3.2, while condition (iii) is similar to condition (ii) there with added terms that can be generated by switches at nonzero velocity. These terms are evaluated on the set of velocities that can be generated in regime 2. Conditions (ii) and (iii) imply only local configuration controllability, which means that we would not necessarily be able to stop after we moved to the desired configuration. However, if condition (i) holds, we can bring the velocity to zero at the final position by performing loops there. The system will be therefore equilibrium controllable.

4 Controllability analysis and planning for a sliding and clamping mechanism

In this section we apply our analysis to a two-link mechanism sliding without friction on a plane. The mechanism consists of two homogeneous bars of unit density and lengths (l_1, l_2), connected by an actuated rotational joint (Figure 2). In the figure, CM denotes the center of mass of the two body system. The coordinates of the center of mass of the link j are (x_j, y_j, θ_j), while $(x_{\text{CM}}, y_{\text{CM}})$ are the coordinates of CM. One of the links can be instantaneously clamped to the ground (anywhere on the plane) so that it gets completely immobilized. When the link is clamped, the number of degrees of freedom of the system decreases from 4 to 1. The clamping constraint is described by the function $\varphi_1(q) = (x_1, y_1, \theta_1)$.

The configuration manifold of the two body system is $Q = SE(2) \times S^1$. We will describe the configuration with the coordinates $q = (x_{\text{CM}}, y_{\text{CM}}, \theta, \phi)$, where $\theta = \theta_1$ and $\phi = \theta_2 - \theta_1$.

When the first link is clamped, the system is confined to the submanifold $R_1(q_0) = \{q \in Q \mid \varphi_1(q) = (x_0, y_0, \theta_0)\}$. This holonomic constraint induces the constraint distribution:

$$\mathcal{D}_2(q) = \text{span}\left\{-2l_2^2\sin(\phi+\theta)\frac{\partial}{\partial x_{\text{CM}}} + 2l_2^2\cos(\phi+\theta)\frac{\partial}{\partial y_{\text{CM}}} + 4(l_1+l_2)\frac{\partial}{\partial\phi}\right\}$$

If we set $\mathcal{D}_1(q) = T_qQ$, we thus have a hybrid mechanical control system:

$$\Sigma = (\{1,2\}, Q, M, \mathcal{F}, U, \{\mathcal{D}_1, \mathcal{D}_2\}).$$

The input vector field is:

$$Y_1^1 = l_2^2\left(l_2(5l_1+2l_2) + 3l_1^2\cos(\phi)\right)\frac{\partial}{\partial\theta} - \left(2l_1^4 + 5l_1^3l_2 + 5l_1l_2^3 + 2l_2^4 + 6l_1^2l_2^2\cos(\phi)\right)\frac{\partial}{\partial\phi}$$

The input vector field on the constrained regime is obtained by projecting Y_1^1 onto the constraint distribution. Since the constrained distribution is one-dimensional, the constrained input spans exactly the same direction as the constrained distribution:

$$Y_1^2 = -2l_2^2\sin(\phi+\theta)\frac{\partial}{\partial x_{\text{CM}}} + 2l_2^2\cos(\phi+\theta)\frac{\partial}{\partial y_{\text{CM}}} + 4(l_1+l_2)\frac{\partial}{\partial\phi}$$

We now have all the necessary tools to check for equilibrium controllability. Notice that in the regime 2 the system is fully controllable, $\text{Sym}(\{Y_1^2\}) = \mathcal{D}$, so the condition (i) is trivially satisfied. We can also show that $\langle Y_1^1 : Y_1^1\rangle = \zeta Y_1^1$ for some scalar function ζ, so that condition (ii) is satisfied. Finally, we can check that

$$\text{rank}\left\{Y_1^1, Y_1^2, [Y_1^1, Y_1^2], \nabla^2 Y_1^1(Y_1^2, Y_1^2) + R(Y_1^2, Y_1^1)Y_1^2\right\}(q) = 4$$

in a neighborhood of the point $(x_{\text{CM}}, y_{\text{CM}}, \theta, \phi) = (0,0,0,0)$. Therefore, the hybrid mechanical control system Σ is equilibrium controllable.

It turns out that we could replace the last vector with the bracket $[Y_1^1, [Y_1^2, Y_1^1]]$ and still span the whole space. This means that the system is equilibrium controllable even if we only allow switches at zero velocity, as was shown in [3]. However, this bracket is of higher order than the bracket generated by the switches at nonzero velocity. Therefore, even if switching at nonzero velocity does not contribute any new controllable directions, it allows us to move in certain directions more efficiently.

The controllability analysis also indicates what switching sequence will be necessary to move (locally) in a particular direction. Clearly, the directions Y_1^1 and Y_1^2 do not require any switches. The direction $[Y_1^1, Y_1^2]$ can be generated by a sequence $UCUC$, where U stands for the unclamped regime and C for the clamped regime. And finally, the direction $\nabla^2 Y_1^1(Y_1^2, Y_1^2) + R(Y_1^2, Y_1^1)Y_1^2$ will require the sequence CUC. In the clamped regime we first generate a non-zero velocity in the direction Y_1^2. After switching to the unclamped regime the system will start moving in the direction $\nabla^2 Y_1^1(Y_1^2, Y_1^2) + R(Y_1^2, Y_1^1)Y_1^2$, but in order to stop the motion we need to switch to the clamped regime again at the end of the maneuver.

At this point a motion planning method such as that described in [16] can be employed to construct a motion plan. Since the controllability analysis only gives us local results, we will in general need to concatenate a series of maneuvers in order to reach the desired goal configuration. Figure 3 shows for example a maneuver that can move the mechanism for a unit distance in the y direction. The mechanism is initially clamped. It then swings the unclamped link to build the velocity, after which it releases the constraint and starts drifting. At the end, a maneuver is performed that stops the mechanism.

The motion plan in the figure was computed using the sequential quadratic programming method provided in the NAG numerical library. For the simulation, we fixed the duration of motion to 3s, with the switches occuring at 1s and 2s. The computed trajectory minimizes the L^2 norm of the input.

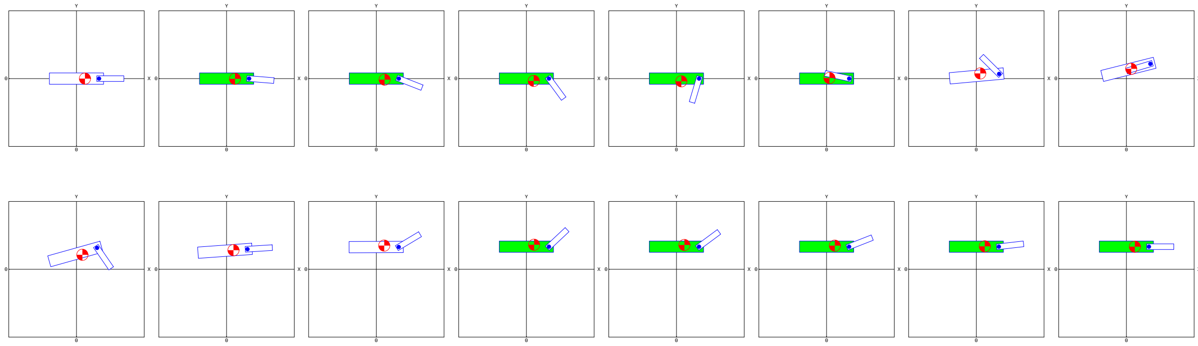

Figure 3: Motion generated by switch at nonzero velocity. When the link is clamped to the ground, it is colored in gray.

5 Conclusions

We investigated a class of mechanisms that can locomote by switching between constraints. The non-smooth nature of such systems due to changes in the dynamic equations and impacts prevents the application of conventional tools for motion planning and control. We showed that under certain conditions it is possible to perform the controllability analysis for such non-smooth locomotion devices by exploiting their special Lagrangian structure. The controllability analysis leads to motion primitives that can be efficiently computed and subsequently used to construct motion plans. We also characterized motion directions in the case when the system switches between constraints at nonzero velocity. We showed that switches at nonzero velocity can provide new controllable directions and can generate motions in certain directions more efficiently than gaits provided by the zero velocity analysis.

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